Simple communication scenario:

One sender Alice

One receiver Bob

Goal: transmit data from Alice to Bob

Example 1:

Data: classical message $m \rightarrow 0, 1$. (1 bit message).

Given: noiseless channel $N$ with input symbol $X = \{0, 1\}$

Output $Y = \{0, 1\}$

$$\text{s.t. } \Pr(Y = y | X = x) = \delta_{yx} \quad \text{if } x \neq y$$

$$1 \quad \text{if } x = y$$

Method: Alice set the input $X = m$, so Bob's output is $m$.

The ability to transmit 1 classical bit: 1 bit can be given or acted

Note: If $m = 0, 1, \ldots, n-1$,

it can be expressed in binary form

and be transmitted to Bob using $\log_2 n$ bits
Example 2:

Data: classical message $a \in \{0,1\}$ as qubit

Given: noiseless quantum channel $N$ transmitting any "q state on 2-dim sys" from Alice to Bob perfectly.

Method: Alice set the input $14m \geq 1m$

where $|0\rangle, |1\rangle$ basis for $C^2$.

Bob measures his output along the basis $\{|0\rangle, |1\rangle\}$

and the outcome is "m".

Call the ability to transmit 1 qubit : qubit

We say 1 qubit $\geq$ 1 cbit

There is a method to

What you are given achieve

What you can start with

Wish it to be

If m has n values, it can be transmitted to Bob using $\log_2 n$ qubits.

Qn : Can we send K bits using n qubits for $K > n$?

Ans : No & Yes. Need to spell out assumptions...
Example 3: Superdense coding SD

Data: m = 0, 1, 2, 3.\textsuperscript{3} (2 bits of data)

Given: 1 qubit

Also Alice & Bob share "1 ebit":

$$|\Psi\rangle_{AB} = \frac{1}{\sqrt{2}} (|10\rangle + |11\rangle)_{AB}$$

Method: Alice affixes \textcolor{red}{m} to system A.

\begin{align*}
\delta_0 &= \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \quad \delta_0 \otimes I_{\text{\textcolor{red}{m}}} = \frac{1}{\sqrt{2}} (|10\rangle + |11\rangle)_{AB} \\
\delta_1 &= \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \quad \delta_1 \otimes I_{\text{\textcolor{red}{m}}} = \frac{1}{\sqrt{2}} (|10\rangle + |10\rangle)_{AB} \\
\delta_3 &= \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \quad \delta_3 \otimes I_{\text{\textcolor{red}{m}}} = \frac{1}{\sqrt{2}} (|10\rangle - |1\rangle)_{AB} \\
\delta_2 &= \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \quad \delta_2 \otimes I_{\text{\textcolor{red}{m}}} = \frac{1}{\sqrt{2}} (|11\rangle - |01\rangle)_{AB}
\end{align*}

With unitary

Mutually orthogonal

so can be applied.

Form in "Bell basis."

(2) Alice sends A to Bob using 1 qbit.

(3) Bob now has both A & B, makes a Bell measurement to find out m.

Resource inequality: 1 qbit + 1 ebit \geq 2 qbits.

Do we cheat, since Alice may have to send B to Bob to create \textcolor{red}{|\Psi\rangle}?

No. \textcolor{red}{|\Psi\rangle} can be created by Bob sending A to Alice.

Or be given to them by a 3rd Party. It's there before m.
Circuit diagram:

- The message, art, and virus at Alice's lab.
- Space and time.
- Boxes = operations.
Example 4: teleportation TP

Data: 1 qubit state $|\psi\rangle = |a\rangle_0 + b|1\rangle_0$

Given: 1 ebit & 2 cbits

(So Alice & Bob start $|\psi\rangle_{AB}$
and Alice can transmit one of 4 qubits to Bob.)

Idea: $|\psi\rangle_A, |\psi\rangle_{AB} = (|a\rangle_0 + b|1\rangle_0) \frac{1}{\sqrt{2}} (|00\rangle + |11\rangle)_{AB}$

$$= \frac{1}{2} \left\{ \frac{1}{\sqrt{2}} \left( |00\rangle + |11\rangle \right)_{AA} \left( |a\rangle_0 + b|1\rangle_0 \right)_{BB} \\
+ \frac{1}{\sqrt{2}} \left( |00\rangle - |11\rangle \right)_{AA} \left( |a\rangle_0 - b|1\rangle_0 \right)_{BB} \\
+ \frac{1}{\sqrt{2}} \left( |01\rangle + |10\rangle \right)_{AA} \left( |a\rangle_1 + b|1\rangle_1 \right)_{BB} \\
+ \frac{1}{\sqrt{2}} \left( |01\rangle - |10\rangle \right)_{AA} \left( |a\rangle_1 - b|1\rangle_1 \right)_{BB} \right\}$$

E.g. $|a\rangle_111>$

Method: ① Alice applies Bell's test on $AA$.
Collapsing the state to one of the 4 lines.
Outcome $m = 0, 1, 2, 3$ respectively.

② Alice sends $m$ to Bob using 2 cbits.

③ Bob now knows he has $6_m|\psi\rangle$.
So he applies $6_m^{-1} = 6_m$ to $B$ to recover $|\psi\rangle$. 
Note:

1. a, b take O many bits to describe
   and give 1 specimen of |x>|. Alice cannot
   extract much of that info (topic 4).

She doesn't need to know a, b and only
needs to send 2 bits to Bob q2 \& m entangled.

2. What does Bob have BEFORE receiving m?
   Each m occurs with prob 1/4 (the norm square
   of each line corresponding is equal).

Bob's state is
\[ \frac{1}{4} |00\rangle |0\rangle |0\rangle + \frac{1}{4} |01\rangle |0\rangle |0\rangle + \frac{1}{4} |10\rangle |0\rangle |0\rangle + \frac{1}{4} |11\rangle |0\rangle |0\rangle. \]

\[ \text{prob density matrix} \]

But it is also half of \( |\psi\rangle \): \[ \text{tr}_A |\psi\rangle \otimes |\psi\rangle = \frac{1}{2}. \]

\[ \text{basis} \rightarrow 00 \ 01 \ 10 \ 11 \]

\[ \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \end{pmatrix} \rightarrow \frac{1}{2} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \frac{1}{2}. \]

So no signalling to 8, 6 until "m" read.
The purple box simulates or achieves the transmission of $|Y\rangle$ from $A_1$ to $B_1$ and consumes 2 ebits & 2 qubits.

i. TP: 2 ebits + 1 ebit = 1 qubit

Ex: What happens if Alice wants to transmit $|X\rangle \in \mathbb{C}^n$?

What about a referee preparing $|\Psi\rangle \in \mathbb{C}^n$ and the goal is to have $|\Psi\rangle_{RB}$ in the end?

ie how to teleport part of an entangled state when Alice & Bob have no access to R?
Density matrix of $B$ before "m" is received is well defined.

\[
\begin{bmatrix}
1 & \frac{1}{4}
\end{bmatrix}
\begin{bmatrix}
\frac{3}{4} & \frac{1}{2} \\
\frac{1}{2} & \frac{1}{2}
\end{bmatrix}
\begin{bmatrix}
1 \\
0
\end{bmatrix}
= \frac{1}{2}.
\]

This gives an encryption scheme.

NB: 147 disappeared after Bell made 6 before W arrives @ B. B.

It is a secret sharing scheme splitting 147 into $m = 6$ and $147 \times 147$ with equal prob.

- Peres the method to perform QEC by entangled purification.

- It's the soul not the body being teleported - Asher Peres.
Quantum information: $|\psi\rangle_5 \in \mathbb{C}^d$, or $\rho \in \mathbb{B}(\mathbb{C}^d)$

Classical information: RV $p(x) = \rho_{x}^c$

Definition of larger Hilbert space: $\rho_S = \frac{1}{d} \sum_{x} |x\rangle \langle x|$

where $|s\rangle = \sum_{x} \langle x| \rho_c \langle x|$.

$\psi_S = \sum_{x} p(x) |x\rangle \otimes |x\rangle_1$

Diagonal density matrix $\rho_{\text{classical}}$

For a basis $|\theta\rangle_R$: $A |\theta\rangle_R \rightarrow 1 |\theta\rangle_R$

For a basis $|\phi\rangle_R$:

$A |\phi\rangle_R \rightarrow |\phi\rangle R_B$

For a basis $|\phi\rangle_R$:

$\forall |\phi\rangle_R, \text{linear}

\forall |\phi\rangle_R, \text{linear}

|\phi\rangle_R$.