Fall 2012 QIC 890 / CO 781 Assignment 7
Due Nov 15, 2012 (in mailbox)

Question 1. An antidegradable depolarizing channel [5 mark]
Show that the depolarizing channel $N(\rho) = \frac{2}{3}\rho + \frac{1}{3}I$ is antidegradable.
(Note the above channel is $N_q(\rho) = (1 - q)\rho + \frac{q}{3}(X\rho X + Y\rho Y + Z\rho Z)$ for $q = 1/4$.)
A particular method is given in the back page.

Question 2. Zero quantum capacity of PPT channels
Let $M$ be a positive semidefinite capacity of PPT channels acting on $\mathbb{C}^{d_A} \otimes \mathbb{C}^{d_B}$.
The partial transpose of $M$ with respect to the second system, denoted by $MT_2$ is defined as follows.
If $M = \sum_{a=1}^{d_A} \sum_{a'=1}^{d_A} \sum_{b=1}^{d_B} \sum_{b'=1}^{d_B} c_{aa'bb'} |a\rangle \langle a'|_A \otimes |b\rangle \langle b'|_B$,
then, $MT_2 = \sum_{a=1}^{d_A} \sum_{a'=1}^{d_A} \sum_{b=1}^{d_B} \sum_{b'=1}^{d_B} c_{aa'bb'} |a\rangle \langle a'|_A \otimes |b\rangle \langle b'|_B$.

Likewise, define $MT_1 = \sum_{a=1}^{d_A} \sum_{a'=1}^{d_A} \sum_{b=1}^{d_B} \sum_{b'=1}^{d_B} c_{aa'bb'} |a\rangle \langle a'|_A \otimes |b\rangle \langle b'|_B$.

Let $|\Phi_d\rangle$ denote the maximally entangled state on two $d$-dimensional systems as usual. Let $\Phi_d = |\Phi_d\rangle \langle \Phi_d|$.
You will show that for any channel $N$, if $I \otimes N(\Phi_d)$ is PPT, then $Q(N) = 0$.

(a) [1 mark] Show that $(MT_1)^T_2 = MT$ where $MT$ is the usual transpose of $M$.

(b) [1 mark] Show that $MT_1$ is positive semidefinite iff $MT_2$ is positive semidefinite.

(c) [2 marks] Show that if $X$ and $Y$ are $d_B \times d_B$ matrices, then, $[(I \otimes X)M(I \otimes Y)]^T_2 = (I \otimes Y^T)MT_2(I \otimes X^T)$.

(d) [2 marks] If $d_A = d_B = d$, show that any state $|\psi\rangle_{RA}$ can be written as $(I \otimes X)|\Phi_d\rangle$ for some $d \times d$ matrix $X$.

(e) [3 marks] Show that if $(I \otimes N)|\Phi_d\rangle$ is PPT, $(I \otimes N)\langle |\psi\rangle|\psi\rangle$ is PPT for any state $|\psi\rangle_{RA}$.

(f) [1 mark] What can you say about $I \otimes N^\otimes n(|\psi\rangle\langle |\psi\rangle)$ for any state $|\psi\rangle_{R_1\ldots R_n A_1\ldots A_n}$?

(g) [2 marks] What is the spectrum of $\Phi_d^{T_2}$?

(h) [3 marks] Show that, if Alice can transmit one qubit to Bob with arbitrary precision using an encoder, $n$ copies of $N$, followed by Bob’s local operation, we get a contradiction. This implies $Q(N) = 0$.

Remark: You have learnt that antidegradable channels have zero quantum capacity. You will see in class that they may communicate quantum data if two-way noiseless classical communication is available for free. (Note this is another form of “superactivation.”) The PPT channels, on the other hand, still have no quantum capacity even if two-way noiseless classical communication is available for free.
Steps for Q1:
First, find an isometric extension $U$ for $N$.
Second, note that it suffices to consider the “Choi-state” with purification, that is, the state $|\alpha\rangle = I \otimes U|\Phi_2\rangle$ which lives in systems $R, B$ and $E$.
Third, find another isometry $V$ that takes $E$ to $E'F$ such that $I_{RB} \otimes V|\alpha\rangle$ is invariant when $B$ and $E'$ are swapped.