Fall 2012 QIC 890 / CO 781 Assignment 7

Due Nov 15, 2012 (in mailbox)

Question 1. An antidegradable depolarizing channel [5 mark]

Show that the depolarizing channel $\mathcal{N}(\rho) = \frac{2}{3}\rho + \frac{1}{3}\frac{I}{2}$ is antidegradable.

(Note the above channel is $\mathcal{N}_q(\rho) = (1-q)\rho + \frac{q}{3}(X\rho X + Y\rho Y + Z\rho Z)$ for $q = 1/4$.)

A particular method is given in the back page.

Question 2. Zero quantum capacity of PPT channels

Let M be a positive semidefinite operator acting on $\mathbb{C}^{d_A} \otimes \mathbb{C}^{d_B}$.

The *partial transpose* of M with respect to the second system, denoted by M^{T_2} is defined as follows.

If $M = \sum_{a=1}^{d_A} \sum_{a'=1}^{d_A} \sum_{b=1}^{d_B} \sum_{b'=1}^{d_B} c_{aa'bb'} |a\rangle\langle a'|_A \otimes |b\rangle\langle b'|_B$,
then, $M^{T_2} = \sum_{a=1}^{d_A} \sum_{a'=1}^{d_A} \sum_{b=1}^{d_B} \sum_{b'=1}^{d_B} c_{aa'bb'} |a\rangle\langle a'|_A \otimes |b'\rangle\langle b|_B$.

In other words, if M is given as a $d_A \times d_A$ matrix, each entry is a $d_B \times d_B$ matrix, then, M^{T_2} results in the transpose of each $d_B \times d_B$ matrix.

Likewise, define $M^{T_1} = \sum_{a=1}^{d_A} \sum_{a'=1}^{d_A} \sum_{b=1}^{d_B} \sum_{b'=1}^{d_B} c_{aa'bb'} |a'\rangle\langle a|_A \otimes |b\rangle\langle b'|_B$.

M is called "PPT" if M^{T_2} is positive semidefinite.

Let $|\Phi_d\rangle$ denote the maximally entangled state on two d-dimensional systems as usual. Let $\Phi_d = |\Phi_d\rangle\langle\Phi_d|$.

You will show that for any channel $\mathcal{N}$, if $\mathcal{I} \otimes \mathcal{N}(\Phi_d)$ is PPT, then $Q(\mathcal{N}) = 0$.

(a) [1 mark] Show that $(M^{T_1})^{T_2} = M^T$ where M^T is the usual transpose of M.

(b) [1 mark] Show that M^{T_1} is positive semidefinite iff M^{T_2} is positive semidefinite.

(c) [2 marks] Show that if X and Y are $d_B \times d_B$ matrices, then, $[(I \otimes X)M(I \otimes Y)]^{T_2} = (I \otimes Y^T)M^{T_2}(I \otimes X^T)$.

(d) [2 marks] If $d_A = d_R = d$, show that any state $|\psi\rangle_{RA}$ can be written as $(I \otimes X)|\Phi_d\rangle$ for some $d \times d$ matrix X.

(e) [3 marks] Show that if $(\mathcal{I} \otimes \mathcal{N})(\Phi_d)$ is PPT, $(\mathcal{I} \otimes \mathcal{N})(|\psi\rangle\langle\psi|)$ is PPT for any state $|\psi\rangle_{RA}$.

(f) [1 mark] What can you say about $\mathcal{I} \otimes \mathcal{N}^{\otimes n}(|\psi\rangle\langle\psi|)$ for any state $|\psi\rangle_{R_1 \cdots R_n A_1 \cdots A_n}$?

(g) [2 marks] What is the spectrum of $\Phi_d^{T_2}$?

(h) [3 marks] Show that, if Alice can transmit one qubit to Bob with arbitrary precision using an encoder, n copies of $\mathcal{N}$, followed by Bob's local operation, we get a contradition. This implies $Q(\mathcal{N}) = 0$.

Remark: You have learnt that antidegradable channels have zero quantum capacity. You will see in class that they may communicate quantum data if two-way noiseless classical communication is available for free. (Note this is another form of "superactivation.") The PPT channels, on the other hand, still have no quantum capacity even if two-way noiseless classical communication is available for free.

Steps for Q1:

First, find an isometric extension U for $\mathcal{N}$.

Second, note that it suffices to consider the "Choi-state" with purification, that is, the state $|\alpha\rangle = I \otimes U|\Phi_2\rangle$ which lives in systems R, B and E.

Third, find another isometry V that takes E to $E'F$ such that $I_{RB} \otimes V|\alpha\rangle$ is invariant when B and E' are swapped.