Fall 2012 QIC 890 / CO 781 Assignment 4

Due Oct 18, 2012 (in class)

Question 1. The capacity of the classical Z channel [5 marks]

The classical Z channel has input alphabet Ω_X equal to the output alphabet Ω_Y equal to $\{0,1\}$. If the input is $x = 0$, the output is $y = 0$ with certainty. If the input is $x = 1$, then the output distribution is uniform over $\{0,1\}$. Find the optimal distribution for the capacity expression and also the capacity for the channel.

Note: it is a one-parameter optimization.

Question 2. Strong additivity of the classical capacity [5 marks]

Let $\mathcal{N}_1$ and $\mathcal{N}_2$ be two arbitrary classical channels. Show that $C(\mathcal{N}_1 \otimes \mathcal{N}_2) = C(\mathcal{N}_1) + C(\mathcal{N}_2)$.

Note 1: For $i = 1,2$, if $\mathcal{N}_i$ takes input alphabet Ω_{X_i} to output alphabet Ω_{Y_i}, then $\mathcal{N}_1 \otimes \mathcal{N}_2$ takes $\Omega_{X_1} \times \Omega_{X_2}$ to output alphabet $\Omega_{Y_1} \times \Omega_{Y_2}$, and $\Pr(Y_1Y_2 = y_1y_2 | X_1X_2 = x_1x_2) = \Pr(Y_1{=}y_1 \mid X_1{=}x_1) \cdot \Pr(Y_2 = y_2 | X_2 = x_2)$.

Note 2: One side is simple.

Note 3: The optimal distribution for the capacity expression of $\mathcal{N}_1 \otimes \mathcal{N}_2$ has very special structure.

Question 3. Some simple exercises on Q boxes [10 marks]

Throughout this question, states are in $\mathbb{C}^2$, and when you are asked to calculate the capacities, derive the optimal distribution as well.

(a) [2 marks] Let I, σ_x, σ_y, σ_z denote the Pauli matrices.

$|\psi_1\rangle\langle\psi_1| = \frac{1}{2}(I + \frac{1}{\sqrt{3}}(\sigma_x + \sigma_y + \sigma_z))$

$|\psi_2\rangle\langle\psi_2| = \frac{1}{2}(I + \frac{1}{\sqrt{3}}(\sigma_x - \sigma_y - \sigma_z))$

$|\psi_3\rangle\langle\psi_3| = \frac{1}{2}(I + \frac{1}{\sqrt{3}}(-\sigma_x + \sigma_y - \sigma_z))$

$|\psi_4\rangle\langle\psi_4| = \frac{1}{2}(I + \frac{1}{\sqrt{3}}(-\sigma_x - \sigma_y + \sigma_z))$

Note that $|\langle\psi_i|\psi_j\rangle|$ is constant for $i \neq j$, and the Bloch vectors of these states form the vertices of a tetrahedron.

(i) What is the pretty good measurement corresponding to these states?

(ii) What is the classical capacity of a Q-box capable of emitting $|\psi_i\rangle$ $(i = 1,2,3,4)$?

(b) [4 marks] Consider the states $\rho_0 = |0\rangle\langle 0|$, $\rho_1 = \frac{1}{2}(\frac{I}{2} + |1\rangle\langle 1|)$.

(i) What is the pretty good measurement corresponding to these states?

(ii) What is the classical capacity of a Q-box capable of emitting ρ_0 and ρ_1?

(c) [4 marks] Consider the states $\rho_0 = |0\rangle\langle 0|$, $\rho_1 = \frac{1}{2}(\frac{I}{2} + |+\rangle\langle +|)$, $\rho_2 = \frac{1}{2}(\frac{I}{2} + |-\rangle\langle -|)$. What is the classical capacity of a Q-box capable of emitting ρ_0, ρ_1, and ρ_2?