Oct 31 notes on SDP for $|||\Phi|||_1$.

Replacing P10 - P14 of the notes for Oct 24.

Notes taken from LN 2011, Cp 21, esp on 21.1 and 21.4

Reading exercise: 21.2 - 21.3.

- Def: (max output fidelity).

Let $\Psi_0, \Psi_1 \in T(\mathcal{X}, \mathcal{Z})$ be positive maps.

Max output fidelity between Ψ_0, Ψ_1

$$= F_{max}(\Psi_0, \Psi_1) = \max \{ F(\Psi_0(\rho_0), \Psi_1(\rho_1)) : \rho_0, \rho_1 \in D(\mathcal{X}) \}$$

allowed to be different

ρ_0, ρ_1 have to be trace 1

- Relating F_{max} & $|||\Phi|||_1$:

Thm 21.1: For $i = 0, 1$

let $A_i \in L(\mathcal{X}, \mathcal{Y} \otimes \mathcal{Z})$

$$\Psi_i(M) = \mathrm{Tr}_{\mathcal{Y}}(A_i M A_i^*) \quad \forall M \in L(\mathcal{X}), \ \Psi_i \in T(\mathcal{X}, \mathcal{Z})$$

$$\Phi(M) = \mathrm{Tr}_{\mathcal{Z}}(A_0 M A_1^*) \quad \cdots \quad \Phi \in T(\mathcal{X}, \mathcal{Y})$$

Then $|||\Phi|||_1 = F_{max}(\Psi_0, \Psi_1)$.

NB: 3 ways to interpret what's given in the hypothesis

① given A_0, A_1, derive Ψ_0, Ψ_1, Φ, then ....

② given Ψ_0, Ψ_1, derive A_0, A_1, Φ, ...

③ given Φ, derive A_0, A_1, Ψ_0, Ψ_1, ...

NB: Our main application has $\Phi = \Phi_0 - \Phi_1$ where $\Phi_{0,1}$ are CP maps.

Ex 1: Show that if $\Phi \in C(\mathcal{X},\mathcal{Y})$, then $A_0 = A_1$ and $f_{max}(\Phi_0, \Phi_1) = 1$, which is consistent with $|||\Phi|||_1 = 1$ from def of $|||\cdot|||_1$.

Ex 2: Let $\Phi(M) = \sum_{k\in\Sigma_1} E_k M E_k^* - \sum_{l\in\Sigma_2} B_l M B_l^*$ s.t. $\Sigma_1 \cap \Sigma_2 = \emptyset$, $B_l, E_k \in L(\mathcal{X},\mathcal{Y})$

Find $A_0, A_1 \in L(\mathcal{X}, \mathcal{Y}\otimes\mathcal{Z})$ s.t. $\mathrm{tr}_{\mathcal{Z}}\, A_0 M A_1^* = \Phi(M)$.

We use Lemma 21.3 to prove Thm 21.1.

<u>Lemma 21.3</u>: Let $u, v \in \mathcal{K}\otimes\mathcal{L}$.

Then $F(\mathrm{Tr}_{\mathcal{L}}\, uu^*, \mathrm{Tr}_{\mathcal{L}}\, vv^*) = \|\mathrm{Tr}_{\mathcal{K}}\, uv^*\|_1$

over $\mathrm{Pos}(\mathcal{K})$ — over $L(\mathcal{L})$

Pf Let $P = \mathrm{Tr}_{\mathcal{L}}\, uu^*$, $Q = \mathrm{Tr}_{\mathcal{L}}\, vv^*$, $P, Q \in \mathrm{Pos}(\mathcal{X})$.

$$F(P,Q) = \max_{W\in U(\mathcal{L})} \langle v, \underbrace{(\mathbb{1}_{\mathcal{K}}\otimes W)u}\rangle$$

fixed purification for Q

optimal purification for P, with W's phase chosen to make $\langle\, ,\, \rangle \geq 0$

$$= \max_{W\in U(\mathcal{L})} \mathrm{tr}_{\mathcal{K}\mathcal{L}}\, (\mathbb{1}_{\mathcal{K}}\otimes W)\, uv^*$$

$$= \max_{W\in U(\mathcal{L})} \mathrm{tr}_{\mathcal{L}} \left[W \cdot \mathrm{tr}_{\mathcal{K}}\, uv^* \right]$$

$$= \|\mathrm{tr}_{\mathcal{K}}\, uv^*\|_1 \quad \text{by char of } \|\cdot\|_1$$

Pf (Thm 21.1):

Denote the unit sphere of a CES as $S(\cdot)$.

$$|||\Phi|||_1 = \max_{a,b \in S(\mathcal{X}\otimes\mathcal{W})} \left\{ \underbrace{\| \Phi \otimes I_{\mathcal{W}}(ab^*) \|_1}_{\text{over } L(\mathcal{Y}\otimes\mathcal{W})} \right\}$$
(by def of $|||\cdot|||_1$ & Lemma 20.2 $\mathcal{W}\sim\mathcal{X}$)

$$= \max_{a,b \in S(\mathcal{X}\otimes\mathcal{W})} \left\{ \| \underbrace{\mathrm{Tr}_{\mathcal{Z}}}_{\mathcal{X}} \underbrace{(A_0 \otimes \mathbb{1}_{\mathcal{W}}) a}_{u} \ \underbrace{b^* (A_1^* \otimes \mathbb{1}_{\mathcal{W}})}_{v^*} \|_1 \right\}$$
expanding Φ

in Lemma 21.3

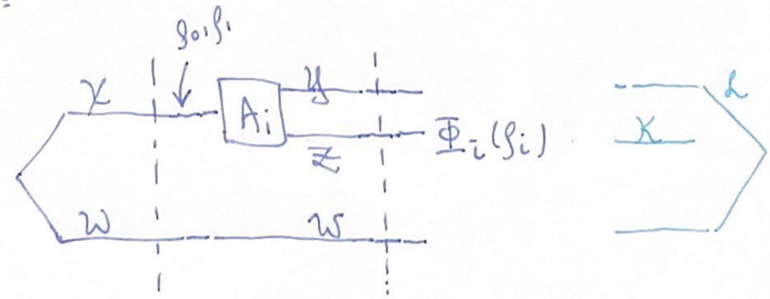
so $\mathcal{L} \leftrightarrow \mathcal{Y}\otimes\mathcal{W}$

$$= \max_{a,b \in S(\mathcal{X}\otimes\mathcal{W})} \left\{ F\left(\mathrm{Tr}_{\mathcal{Y}\otimes\mathcal{W}} (A_0 \otimes \mathbb{1}_{\mathcal{W}}) a a^* (A_0^* \otimes \mathbb{1}_{\mathcal{W}}), \ \mathrm{Tr}_{\mathcal{Y}\otimes\mathcal{W}} (A_1 \otimes \mathbb{1}_{\mathcal{W}}) b b^* (A_1^* \otimes \mathbb{1}_{\mathcal{W}}) \right) \right\}$$

$$= \max_{a,b \in S(\mathcal{X}\otimes\mathcal{W})} \left\{ F\left(\underset{\uparrow \, \mathrm{Tr}_{\mathcal{Y}} A_0 \cdot A_0^*}{\Psi_0} (\underbrace{\mathrm{Tr}_{\mathcal{W}}(aa^*)}_{\rho_0}), \ \underset{\uparrow \, \mathrm{Tr}_{\mathcal{Y}} A_1 \cdot A_1^*}{\Psi_1} (\underbrace{\mathrm{Tr}_{\mathcal{W}}(bb^*)}_{\rho_1}) \right) \right\}$$

$$= \max_{\rho_0, \rho_1 \in D(\mathcal{X})} F(\Psi_0(\rho_0), \Psi_1(\rho_1)) = F_{\max}(\Psi_0, \Psi_1).$$

Diagram:



$a, b \in S(\mathcal{X}\otimes\mathcal{W})$ $\quad u, v \in \mathcal{Y}\otimes\mathcal{Z}\otimes\mathcal{W}$

Φ is not physical, so cannot draw it....

////////// SDP for $\| \cdot \|_\diamond$: let Φ, Ψ_0, Ψ_1 be given as in Thm 21.1

Define an SDP with var
$$X = \begin{bmatrix} X_0 & & & \\ & X_1 & & \\ & & Z_0 & M \\ & & M^* & Z_1 \end{bmatrix} \in L(\mathcal{X} \oplus \mathcal{X} \oplus \mathcal{Z} \oplus \mathcal{Z})$$
(do not care unfilled blocks)

(actual variables : $\overset{X_0}{\overset{\shortparallel}{\rho_0}}, \overset{X_1}{\overset{\shortparallel}{\rho_1}}, M$)

$$\Xi(X) = \begin{bmatrix} \mathrm{Tr}(X_0) & & & \\ & \mathrm{Tr}(X_1) & & \\ & & Z_0 - \Psi_0(X_0) & \\ & & & Z_1 - \Psi_1(X_1) \end{bmatrix} \in L(\mathbb{C} \oplus \mathbb{C} \oplus \mathcal{Z} \oplus \mathcal{Z})$$
(0's in unfilled blocks)

$$A = \frac{1}{2} \begin{bmatrix} 0 & & & \\ & 0 & & \\ & & 0 & \mathbb{1}_\mathcal{Z} \\ & & \mathbb{1}_\mathcal{Z} & 0 \end{bmatrix} \in L(\mathcal{X} \oplus \mathcal{X} \oplus \mathcal{Z} \oplus \mathcal{Z})$$
(0's in unfilled blocks).

$$B = \begin{bmatrix} 1 & & & \\ & 1 & & \\ & & 0 & \\ & & & 0 \end{bmatrix} \in L(\mathbb{C} \oplus \mathbb{C} \oplus \mathcal{Z} \oplus \mathcal{Z})$$
(0's in unfilled blocks)

Primal SDP = $\sup \langle A, X \rangle$
s.t. $\Xi(X) = B$
$X \geq 0$

Part ① will show this is $|||\Phi|||_1$ by showing it is $F_{max}(\Psi_0, \Psi_1)$

② Then will invoke Slater's Thm

Dual = $\inf \langle B, Y \rangle$
s.t. $\Xi^*(Y) \geq A$
$Y \in \mathrm{Herm}(\mathbb{C} \oplus \mathbb{C} \oplus \mathcal{X} \oplus \mathcal{X})$.

Part ② this will be our char of $\| \cdot \|_\diamond$

Let
$$Y = \begin{bmatrix} y_0 & & & \\ & y_1 & & \\ & & Y_0 & \\ & & & Y_1 \end{bmatrix}$$
don't care about unfilled blocks.

Part ①

- Rephrasing the primal constraints:

① $X_0, X_1, Z_0, Z_1 \geq 0$, $\begin{bmatrix} Z_0 & M \\ M^* & Z_1 \end{bmatrix} \geq 0$

② $\operatorname{tr} X_0 = \operatorname{tr} X_1 = 1$ (first 2 blocks of B)

③ $Z_0 = \Psi_0(X_0)$, $Z_1 = \Psi_1(X_1)$ (last 2 blocks of B) $X_i \leftrightarrow \rho_i$

Note ① gives $X \geq 0$ with 0 unfilled blocks.

∴ values determined by ① can be attained

Meanwhile, general $X \geq 0$ has blocks satisfying ① but with more constraints.

Such X cannot increase the value of the SDP, beyond those with 0 unfilled blocks.

- Obj fcn $= \frac{1}{2}(\operatorname{tr} M + \operatorname{tr} M^*)$

∴ Primal SDP equiv to

$$\sup \ \frac{1}{2}(\operatorname{tr} M + \operatorname{tr} M^*)$$

$$\text{s.t.} \begin{bmatrix} \Psi_0(X_0) & M \\ M^* & \Psi_1(X_1) \end{bmatrix} \geq 0$$

$$X_0, X_1 \in D(\mathcal{X})$$

~~∵ Fmax(Ψ0, Ψ1) ∃ X1, X2 optimizing two~~

$$\text{Optimal value} = \sup_{X_0, X_1 \in D(X)} F(\Psi_0(X_0), \Psi_1(X_1))$$ by SDP char of fidelity

$$= F_{max}(\Psi_0, \Psi_1)$$

$$= |||\Psi|||_1 \text{ by Thm 21.1}$$

Part 2

For the dual:

$$\langle \Xi^*(Y), X \rangle = \langle Y, \Xi(X) \rangle$$

$$= \left\langle \begin{pmatrix} y_0 & & & \\ & y_1 & & \\ & & Y_0 & \\ & & & Y_1 \end{pmatrix}, \begin{pmatrix} \operatorname{Tr} X_0 & 0 & 0 & 0 \\ 0 & \operatorname{Tr} X_1 & 0 & 0 \\ 0 & 0 & Z_0 - \Psi_0(X_0) & 0 \\ 0 & 0 & 0 & Z_1 - \Psi_1(X_1) \end{pmatrix} \right\rangle \quad \text{in } L(\mathbb{C} \oplus \mathbb{C} \oplus Z \oplus Z)$$

$$= \left\langle \begin{pmatrix} y_0 \mathbb{1}_X & 0 & 0 & 0 \\ 0 & y_1 \mathbb{1}_X & 0 & 0 \\ 0 & 0 & Y_0 & 0 \\ 0 & 0 & 0 & Y_1 \end{pmatrix} + \begin{pmatrix} -\Psi_0^*(Y_0) & & & 0 \\ & -\Psi_1^*(Y_1) & & \\ & & 0 & \\ 0 & & & 0 \end{pmatrix}, \begin{pmatrix} X_0 & & & \\ & X_1 & & \\ & & Z_0 & \\ & & & Z_1 \end{pmatrix} \right\rangle$$

in $L(X \oplus X \oplus Z \oplus Z)$

$$\therefore \Xi^*(Y) = \begin{pmatrix} y_0 \mathbb{1}_X - \Psi_0^*(Y_0) & & & 0 \\ & y_1 \mathbb{1}_X - \Psi_1^*(Y_1) & & \\ & & Y_0 & \\ 0 & & & Y_1 \end{pmatrix}$$

$$\Xi^*(Y) \geq A = \begin{bmatrix} 0 & & & \\ & 0 & & \\ & & 0 & \mathbb{1}_Z \\ & & \mathbb{1}_Z & 0 \end{bmatrix} \frac{1}{2}$$

$$\Leftrightarrow \left.\begin{array}{l} y_0 \mathbb{1}_X \geq \Psi_0^*(Y_0) \\ y_1 \mathbb{1}_X \geq \Psi_1^*(Y_1) \end{array}\right\} (\#)$$

rescaled vars — fun facts for $\begin{bmatrix} Y & -\mathbb{1} \\ -\mathbb{1} & Y' \end{bmatrix} \geq 0$

$$\begin{bmatrix} Y_0 & -\frac{1}{2}\mathbb{1}_Z \\ -\frac{1}{2}\mathbb{1}_Z & Y_1 \end{bmatrix} \geq 0 \quad \Leftrightarrow \quad \boxed{\begin{array}{l} \frac{1}{2} Y = Y_0 \\ \frac{1}{2} Y' = Y_1 \end{array}} \; \boxed{\begin{array}{l} Y \in \mathrm{Pd}(Z) \\ Y' \geq Y^{-1} \end{array}}$$

$$\text{obj fcn} = \inf \langle B, Y \rangle$$

$$= y_0 + y_1 \qquad \text{(by the form of B)}$$

from (#) char of $\| \|_\infty$

$$= \| \Psi_0^*(Y_0) \|_\infty + \| \Psi_1^*(Y_1) \|_\infty$$

$$= \frac{1}{2} \| \Psi_0^*(Y) \|_\infty + \underbrace{\frac{1}{2} \| \Psi_1^*(Y') \|_\infty}$$

$\because \Psi_1^*$ positive map

this is non increasing

if we change $Y' \geq Y^{-1}$ to $Y' = Y^{-1}$.

$$= \frac{1}{2} \| \Psi_0^*(Y) \|_\infty + \frac{1}{2} \| \Psi_1^*(Y^{-1}) \|_\infty$$

$$\therefore \text{Dual} = \inf_{Y \in \mathrm{Pd}(X)} \frac{1}{2} \| \Psi_0^*(Y) \|_\infty + \frac{1}{2} \| \Psi_1^*(Y^{-1}) \|_\infty$$

~~Part (2)~~

Part (3): Strong duality

- Take $X_i \in D(\mathcal{X})$, $Z_i = \Psi_i(X_i) \in Pos(\mathcal{Z})$, $M = 0$ (so $X \geq 0$)
  then $\Xi(X) = B$ ∴ primal SDP is feasible ∴ β bounded from below.
- Take $Y_0 = Y_1 = \mathbb{1}_{\mathcal{Z}}$ (lower block of $\Xi^*(Y) = \begin{bmatrix} \mathbb{1}_{\mathcal{Z}} & 0 \\ 0 & \mathbb{1}_{\mathcal{Z}} \end{bmatrix} > \frac{1}{2}\begin{bmatrix} 0 & \mathbb{1}_{\mathcal{Z}} \\ \mathbb{1}_{\mathcal{Z}} & 0 \end{bmatrix}$)

Want $y_i \mathbb{1}_{\mathcal{X}} - \Psi_i^*(Y_i) > 0$

Take $y_i = \|\Psi_i^*(\mathbb{1}_{\mathcal{Z}})\|_\infty + 1$.

$$\left.\begin{array}{l} \therefore \text{ Dual is strictly feasible.} \\ \therefore\ \beta \text{ bounded above } \therefore\ \beta \text{ finite.} \end{array}\right\} \Rightarrow \text{ST holds} \Rightarrow \alpha = \beta$$

$$\therefore\ |||\Phi|||_1 = \inf_{Y \in Pd(\mathcal{X})} \frac{1}{2}\|\Psi_0^*(Y)\|_\infty + \frac{1}{2}\|\Psi_1^*(Y^{-1})\|_\infty$$

$\mathrm{tr}_{\mathcal{Z}} A_0 \cdot A_1^*$ $\qquad$ $\mathrm{tr}_{\mathcal{Y}} A_0 \cdot A_0^*$ $\qquad$ $\mathrm{tr}_{\mathcal{Y}} A_1 \cdot A_1^*$