QIC 820 / CO781 / CO486 / CS 867

Supplementary notes on SDP for Oct 5 lecture.

① Instead of taking the primal & dual SDP as given, and derive weak duality, one can instead use a Lagrange multiplier method to use weak duality to derive the dual SDP.

For the primal SDP:

$$\begin{aligned} \alpha = \sup\ & \langle A, X \rangle \\ \text{s.t.}\ & \Phi(X) = B \\ & X \geq 0 \end{aligned}$$

associate a dual variable for each primal constraint:

$\Phi(x) = B$ $\longleftrightarrow$ $Y \in \text{Herm}(\mathcal{Y})$

equality constraint — free (Hermitian)

$X \geq 0$ $\longleftrightarrow$ $S \in \text{Pos}(\mathcal{X})$

inequality — positive semidefinite

Then multiplying the pair in each row and summing gives a nonnegative quantity:

$$\langle \underbrace{B - \Phi(X)}_{\text{zero if feasible}}, \underbrace{Y}_{\text{Herm}} \rangle + \langle \underset{\substack{\vee\!\!|\\ 0}}{X}, \underset{\substack{\vee\!\!|\\ 0}}{S} \rangle \geq 0 \quad \text{if feasible}$$

Rearranging: $\underbrace{\langle B, Y \rangle}_{\substack{\text{make this}\\ \text{the objective}\\ \text{function in dual}}} - \langle X, \underbrace{\Phi^*(Y) - S}_{\text{make this } A \text{ in dual}} \rangle \geq 0$

then $\langle B, Y \rangle \geq \langle A, X \rangle$ for any feasible Y, X.

Thus dual:

$$\begin{aligned} \inf\ & \langle B, Y \rangle \\ \text{s.t.}\ & \Phi^*(Y) - S = A \\ & S \geq 0,\ Y \in \text{Herm}(\mathcal{Y}) \end{aligned}$$

(2) Two methods to find $\Phi^*(Y)$ given $\Phi(X)$:

(i) Write $\Phi(X)$ in Kraus rep:

$$\Phi(X) = \sum_k M_k X M_k^* \; s_k$$

where $M_k \in L(X, Y)$, $s_k \in \{\pm 1\}$.

The Kraus rep can always be obtained by the Choi rep:

$$J(\Phi) = \Phi \otimes I(\beta\beta^*), \quad \beta = \sum_a e_a \otimes e_a, \quad \{e_a\}_a \text{ basis for } X$$

followed by spectral decomposition:

$$J(\Phi) = \sum_k \underbrace{\underset{\pm 1}{s_k} \cdot |\lambda_k|}_{\text{eigenvalue of } J(\Phi)} \cdot \underset{\text{eigenvector in } Y \otimes X}{u_k} u_k^*,$$

NB $J(\Phi) \in \mathrm{Herm}(Y \otimes X)$ $\forall \Phi$ Hermiticity preserving

From lecture on characterizations of Q channels,

$$J(\Phi) = \sum_k s_k \, \mathrm{vec}(M_k) \, \mathrm{vec}(M_k)^*$$

so take $M_k = \mathrm{vec}^{-1}(u_k \cdot \sqrt{|\lambda_k|})$ will do.

* But !!! it is often easier to find the M_k's by inspection !!

(ii) Use the def: $\forall X \in L(X)$, $\forall Y \in L(Y)$

$$\langle Y, \underbrace{\Phi(X)}\rangle = \langle \Phi^*(Y), X \rangle$$

(i) you know this from the given $\Phi(x)$

(ii) perform inner product on LHS

(iii) rearrange to a form $\langle \ldots\ldots, X \rangle$ as in RHS, then $\ldots\ldots$ must be $\Phi^*(Y)$.

Demonstrating these ideas in the example:

Primal SDP:

$$\alpha = \sup(-t)$$

$$\text{s.t.} \begin{bmatrix} t & a & b \\ a & 0 & \frac{1-t}{2} \\ b & \frac{1-t}{2} & c \end{bmatrix} \geq 0$$

Claim: Dual SDP

$$\beta = \inf y$$

$$\text{s.t.} \begin{bmatrix} y+1 & 0 & 0 \\ 0 & z & y \\ 0 & y & z \end{bmatrix} \geq 0$$

To verify the dual:

① take $X = \begin{bmatrix} t & a & b \\ a & e & d \\ b & d & c \end{bmatrix}$ most general symmetric 3x3 matrix

as primal variable.

② take the constraints $e=0$, $t+2d=1$

we can define $\Phi(X) = \begin{bmatrix} X_{11}+X_{23}+X_{32} & 0 & 0 \\ 0 & X_{22} & 0 \\ 0 & 0 & 0 \end{bmatrix}$ $(X_{ab} = X(a,b))$

$$B = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad A = \begin{bmatrix} -1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

so that $\alpha = \langle A, X \rangle$ is indeed the primal SDP stated above.

$\text{s.t. } \Phi(X) = B$

$X \geq 0$

To get $\Phi^*(Y)$ using method (i):

One can get the Kraus rep by inspection:

$$\Phi(X) = \sum_{i=1}^{3} M_i X M_i^* - M_4 X M_4^*$$

where $M_1 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ (this gives X_{11} in the (1,1)-entry of $\Phi(X)$)

$M_2 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ (this gives X_{22} in the (2,2)-entry of $\Phi(X)$)

$M_3 = \frac{1}{\sqrt{2}}\begin{bmatrix} 0 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ (this gives $\frac{1}{2}(X_{22} + X_{32} + X_{23} + X_{33})$ in the (1,1)-entry of $\Phi(X)$)

$M_4 = \frac{1}{\sqrt{2}}\begin{bmatrix} 0 & 1 & -1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ (this gives $\frac{1}{2}(X_{22} - X_{32} - X_{23} + X_{33})$ in the (1,1)-entry of $\Phi(X)$)

So $\Phi^*(Y) = \sum_{i=1}^{3} M_i^* Y M_i - M_4^* Y M_4$.

$i=1$ term $= \begin{bmatrix} Y_{11} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$, $i=2$ term $= \begin{bmatrix} 0 & 0 & 0 \\ 0 & Y_{22} & 0 \\ 0 & 0 & 0 \end{bmatrix}$

$i=3$ term $= \frac{1}{2}\begin{bmatrix} 0 & 0 & 0 \\ 0 & Y_{11} & Y_{11} \\ 0 & Y_{11} & Y_{11} \end{bmatrix}$, $i=4$ term $M_4^* Y M_4 = \frac{1}{2}\begin{bmatrix} 0 & 0 & 0 \\ 0 & Y_{11} & -Y_{11} \\ 0 & -Y_{11} & Y_{11} \end{bmatrix}$

$$\therefore \Phi^*(Y) = \begin{bmatrix} Y_{11} & 0 & 0 \\ 0 & Y_{22} & Y_{11} \\ 0 & Y_{11} & 0 \end{bmatrix}$$

Deriving $\Phi^*(Y)$ using method (ii):

$\forall X, Y, \quad \langle \Phi^*(Y), X \rangle = \langle Y, \Phi(X) \rangle$

$$= \left\langle \begin{bmatrix} Y_{11} & Y_{12} & Y_{13} \\ Y_{21} & Y_{22} & Y_{23} \\ Y_{31} & Y_{32} & Y_{33} \end{bmatrix}, \begin{bmatrix} X_{11}+X_{23}+X_{32} & 0 & 0 \\ 0 & X_{22} & 0 \\ 0 & 0 & 0 \end{bmatrix} \right\rangle$$

(NB: I did NOT check which is Y_{ab} vs Y_{ba} $\because$ Y is symmetric.)

$$= \underline{\underline{Y_{11}}}\, X_{11} + \underline{\underline{Y_{11}}}\, X_{23} + \underline{\underline{Y_{11}}}\, X_{32} + \underline{\underline{Y_{22}}}\, X_{22}$$

if this were to be $\langle M, X \rangle$ for some matrix M,

this coeff in front of X_{11} has to be M_{11}

" - - - - - X_{23} - - - M_{23}

" - - - X_{32} M_{32}

" - - X_{22} M_{22}

$\because$ we consider $\mathbb{R}^3$ and symmetric matrices no need to worry about complex conjugates.

$$= \left\langle \begin{bmatrix} Y_{11} & 0 & 0 \\ 0 & Y_{22} & Y_{11} \\ 0 & Y_{11} & 0 \end{bmatrix}, \begin{bmatrix} X_{11} & X_{12} & X_{13} \\ X_{21} & X_{22} & X_{23} \\ X_{31} & X_{32} & X_{33} \end{bmatrix} \right\rangle$$

$$\therefore \quad \Phi^*(Y) = \begin{bmatrix} Y_{11} & 0 & 0 \\ 0 & Y_{22} & Y_{11} \\ 0 & Y_{11} & 0 \end{bmatrix}.$$

2. The dual is:

$$\inf \langle B, Y \rangle$$
$$\text{s.t. } \Phi^*(Y) \succeq A = \begin{bmatrix} -1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Let $Y_{11} = y$, $Y_{22} = z$, $\langle B, Y \rangle = y$, $\Phi^*(Y) = \begin{bmatrix} y & 0 & 0 \\ 0 & z & y \\ 0 & y & 0 \end{bmatrix}$

the dual is

$$\inf\ y$$
$$\text{s.t. } \begin{bmatrix} y+1 & 0 & 0 \\ 0 & z & y \\ 0 & y & 0 \end{bmatrix} \succeq 0$$

(the +1 is from A)

This primal & dual SDP pair satisfies weak duality, finite, but <u>not</u> strong duality as $\alpha \neq \beta$!!