QIC 820 Part 1 lecture 6.

Representations of Q channels. LN 2011 Sec 5.2–5.3, book 2.2.2

① Recall $T(\mathcal{X},\mathcal{Y}) = L(L(\mathcal{X}), L(\mathcal{Y}))$

ⓐ Def (linear rep):

Let $\{A_a\}$ be on basis for $L(\mathcal{X})$, $a \in \Sigma_1$
$\{B_b\}$ ... $L(\mathcal{Y})$, $b \in \Sigma_2$

Then $\forall \Phi \in T(\mathcal{X},\mathcal{Y})$, $\exists!$ matrix rep in $M_{\Sigma_2 \times \Sigma_1}$.

eg. Qubit depolarizing channel $\Phi(\rho) = \frac{I}{2}$ $\forall \rho \in D(\mathcal{X})$.

$\{A_a\} = \{\underset{\mathbb{1}}{\sigma_0}, \underset{x}{\sigma_1}, \underset{y}{\sigma_2}, \underset{z}{\sigma_3}\} = \{B_b\}$

Matrix rep $= \begin{bmatrix} 1 & & & \\ & 0 & & \\ & & 0 & \\ & & & 0 \end{bmatrix}$. $(\Phi(\mathbb{1}) = \mathbb{1},\ \Phi(\sigma_i) = 0\ \forall i = 1,2,3)$.

ⓑ Def (Natural rep):

$\forall \Phi \in T(\mathcal{X},\mathcal{Y})$, $K(\Phi) \in L(\mathcal{X}\otimes\mathcal{X}, \mathcal{Y}\otimes\mathcal{Y})$ satisfies:

$\forall A \in L(\mathcal{X})$, $K(\Phi)\,\mathrm{vec}(A) = \mathrm{vec}(\Phi(A))$.

NB This uses standard bases $\{e_a\} \subseteq \mathcal{X}$, $\{e_b\} \subseteq \mathcal{Y}$ & vec to pick bases for $L(\mathcal{X})$ and $L(\mathcal{Y})$ for the lin rep in ⓐ.

ⓒ Def (Choi-rep):

Let $\beta = \sum_{a\in\Gamma} e_a \otimes e_a$, $\mathcal{X} = \mathbb{C}^\Gamma$.

$\forall \Phi \in T(\mathcal{X},\mathcal{Y})$, $J(\Phi) = \Phi \otimes I_{\mathcal{X}}(\beta\beta^*) \in L(\mathcal{Y}\otimes\mathcal{X}, \mathcal{Y}\otimes\mathcal{X})$

Fact: $J(\Phi) = \sum_{a,b\in\Gamma} \Phi(e_a e_b^*) \otimes e_a e_b^*$

Def: Choi rank of Φ = $\mathrm{rank}(J(\Phi))$

Observations:

(i) $\Phi \mapsto K(\Phi)$ linear, injective, surjective.

(ii) $\Phi \mapsto J(\Phi)$ linear.

To see it is injective, if $J(\Phi_1) = J(\Phi_2)$

$$\sum_{a,b} \Phi_1(|a\rangle\langle b|) \otimes |a\rangle\langle b| \qquad \sum_{a,b} \Phi_2(|a\rangle\langle b|) \otimes |a\rangle\langle b|$$

then $\forall a,b$ $\Phi_1(|a\rangle\langle b|) = \Phi_2(|a\rangle\langle b|)$

$\therefore \Phi_1 = \Phi_2$.

Will see J is surjective later.

(iii) K, J are called "unique" representations

It means K, J are well-defined functions

(iv) Obtaining Φ from $J(\Phi)$:

$\forall A \in L(X)$, let $A = \sum_{a,b} A(a,b)\, |a\rangle\langle b|$, $\Phi(A) = \sum_{a,b} A(a,b)\, \Phi(|a\rangle\langle b|)$

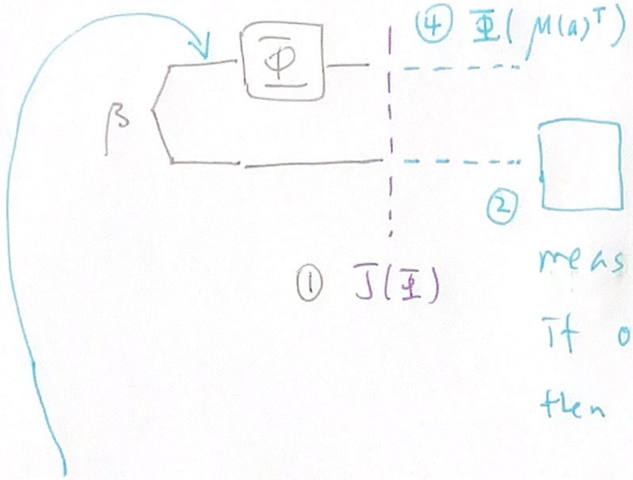
Compared with $J(\Phi) = \sum_{a,b} \Phi(|a\rangle\langle b|) \otimes |a\rangle\langle b|$

$$\mathrm{tr}_X \left[\mathbb{1}_Y \otimes |b'\rangle\langle a'|\, A(a',b') \right] J(\Phi) = \Phi(|a'\rangle\langle b'|)\, A(a',b')$$

sum a', b':

$$\mathrm{tr}_X \left(\mathbb{1}_Y \otimes A^T \right) J(\Phi) = \Phi(A)$$

Alternative derivation $J(\Phi) \to \Phi$:



meas $\{M_a\}$

If outcome $= a$

then unmeasured system of $\beta \propto M_a^T$

③ $M(a)^T$ if outcome $= a$

$$\therefore \operatorname{tr}_X \left(\mathbb{1}_Y \otimes M(a) \right) J(\Phi) = \Phi(M(a)^T).$$

By linearity of transpose & above equation can replace $M(a)^T$ by any $A \in L(X)$.

② 4 reps for $\Phi \in T(\mathcal{X}, \mathcal{Y})$

<u>Prop 5.2</u>: Let $\Phi \in T(\mathcal{X}, \mathcal{Y})$, Γ finite non-empty set $\{A_a : a \in \Gamma\}, \{B_a : a \in \Gamma\} \subseteq L(\mathcal{X}, \mathcal{Y})$. Then:

(i) $K(\Phi) = \sum_{a \in \Gamma} A_a \otimes \overline{B_a}$

$\Leftrightarrow$ (ii) $J(\Phi) = \sum_{a \in \Gamma} \mathrm{vec}(A_a)\, \mathrm{vec}(B_a)^*$

$\Leftrightarrow$ (iii) $\Phi(M) = \sum_{a \in \Gamma} A_a M B_a^*$ (Kraus rep)

$\Leftrightarrow$ (iv) $\Phi(M) = \mathrm{Tr}_{\mathcal{Z}}(A X B^*)$ for $\mathcal{Z} = \mathbb{C}^\Gamma$

$A = \sum_{a \in \Gamma} A_a \otimes e_a \in L(\mathcal{X}, \mathcal{Y} \otimes \mathcal{Z})$

(Stinespring rep)

$B = \sum_{a \in \Gamma} B_a \otimes e_a \in$ "

Pf:

$$\begin{tikzcd}
 & \text{(iii)} \arrow[dl, Leftrightarrow] \arrow[d, Leftrightarrow] \arrow[dr, Leftrightarrow, "\text{elementary, see Nielsen \& Chuang}"] & \\
\text{(i)} & \text{(ii)} & \text{(iv)}
\end{tikzcd}$$

(iii) $\Rightarrow$ (i): $\Phi(M) = \sum_{a \in \Gamma} A_a M B_a^*$

(rec bijection) $\Leftrightarrow$ $\mathrm{vec}(\Phi(M)) = \sum_{a \in \Gamma} \mathrm{vec}(A_a M B_a^*)$ (vec lin !!)

$\Leftrightarrow$ $\mathrm{vec}(\Phi(M)) = \sum_{a \in \Gamma} (A_a \otimes \overline{B_a})\, \mathrm{vec}(M)$ (A1)

$\Leftrightarrow$ $K(\Phi) = \sum_{a \in \Gamma} A_a \otimes \overline{B_a}$. (need uniqueness of K)

(i) $\Rightarrow$ (iii)

Recall $\mathrm{vec}(M) = (M \otimes \mathbb{1}_X)\beta$.

Given (ii), $J(\Phi) = \sum_{a\in\Gamma} \mathrm{vec}(A_a)\,\mathrm{vec}(B_a)^*$ ☆

$= \sum_{a\in\Gamma} (A_a \otimes \mathbb{1}_X)\,\beta\beta^*\,(B_a^* \otimes \mathbb{1}_X)$ ‡

$= \Psi \otimes I_X(\beta\beta^*)$ where $\Psi(M) = \sum_{a\in\Gamma} A_a M B_a^*$.

$= J(\Psi)$

$\because J$ injective $\therefore \Phi = \Psi$.

$\therefore$ (ii) $\Rightarrow$ (iii)

Given (iii), with $\Phi(M) = \sum_{a\in\Gamma} A_a M B_a^*$

$J(\Phi) = \Phi \otimes I_X(\beta\beta^*) =$ ‡ $=$ ☆ $\therefore$ (ii) holds.

3 Observations:

(i) J is surjective.

For any $H \in L(\mathcal{Y}\otimes\mathcal{X}, \mathcal{Y}\otimes\mathcal{X})$

Take SVD $H = \sum_{i=1}^{r} s_i a_i b_i^*$, $r = \mathrm{rank}(H)$, $a_i, b_i \in \mathcal{X}\otimes\mathcal{Y}$

Recall that $a_i = \mathrm{vec}(A_i)$ for some $A_i \in L(\mathcal{X},\mathcal{Y})$

$b_i = \mathrm{vec}(B_i)$ for some $B_i \in L(\mathcal{X},\mathcal{Y})$

$\therefore H = \sum_{i=1}^{r} \mathrm{vec}(s_i A_i)\,\mathrm{vec}(B_i)^* = J(\Phi)$

for some Φ.

(ii) Likewise, the Kraus rep & Stinespring rep each covers all of $T(\mathcal{X},\mathcal{Y})$.

4 Sec 5.3.1: Characterizations of completely positive lin maps

Recall: $\Phi \in T(X,Y)$ is said to be positive if "$P \geq 0 \Rightarrow \Phi(P) \geq 0$",

Φ is completely positive (CP) $\Phi \otimes I_Z$ is positive for all CES Z.

Thm 5.3. $\forall \Phi \in T(X,Y)$, TFAE (the following are equiv).

$\boxed{1}$ Φ CP

$\boxed{2}$ $\Phi \otimes I_X$ positive

$\boxed{3}$ $J(\Phi) \in \mathrm{Pos}(Y \otimes X)$

$\boxed{4}$ $\forall M \in L(X)$, $\Phi(M) = \sum_{a \in \Gamma} A_a M A_a^{+}$

for some finite Γ and $\{A_a : a \in \Gamma\} \subseteq L(X,Y)$

$\boxed{5}$ $\boxed{4}$ holds for $|\Gamma| = \mathrm{rank}(J(\Phi))$

$\boxed{6}$ $\forall M \in L(X)$, $\Phi(M) = \mathrm{tr}_Z\, A M A^*$

for some CES Z and $A \in L(X, Y \otimes Z)$.

$\boxed{7}$ $\boxed{6}$ holds for $\dim(Z) = \mathrm{rank}(J(\Phi))$

$\uparrow$ means

$\exists$ CES Z, $\exists A \in L(X, Y \otimes Z)$

s.t. $\forall M \in L(X)$, $\Phi(M) = \mathrm{tr}_Z\, A X A^*$

Pf structure:

$$\begin{tikzcd}
\boxed{1} \arrow[r, Rightarrow] & \boxed{2} \arrow[r, Rightarrow] & \boxed{3} \arrow[r, Rightarrow] & \boxed{5} \arrow[r, Rightarrow] \arrow[d, Rightarrow] & \boxed{4} \arrow[d, Rightarrow] \\
 & & & \boxed{7} \arrow[r, Rightarrow] & \boxed{6} \arrow[llllu, bend left, Rightarrow]
\end{tikzcd}$$

$\boxed{1} \Rightarrow \boxed{2}$: definition of CP

$\boxed{2} \Rightarrow \boxed{3}$: $\beta\beta^* \geq 0$, $\Phi \otimes I$ positive $\therefore$ $\underbrace{\Phi \otimes I(\beta\beta^*)}_{\overset{\shortparallel}{J(\Phi)}} \geq 0$

$\boxed{3} \Rightarrow \boxed{5}$ $J(\Phi) \in \mathrm{Pos}(\mathcal{Y} \otimes \mathcal{X})$

Similar to (ii)$\Rightarrow$(iii) in prop 5.2.

$$\therefore J(\Phi) = \sum_{i=1}^{r} u_i u_i^* \lambda_i \quad \left(\text{spectral decomp}, \ u_i \in \mathcal{Y} \otimes \mathcal{X}, \ \lambda_i > 0\right)$$

$$= \sum_{i=1}^{r} \mathrm{vec}(A_i)\, \mathrm{vec}(A_i)^*, \qquad u_i = \frac{1}{\sqrt{\lambda_i}} \mathrm{vec}(A_i)$$

$$= \sum_{i=1}^{r} (A_i \otimes \mathbb{1}_{\mathcal{X}}) \beta\beta^* (A_i \otimes \mathbb{1}_{\mathcal{X}})^*$$

$$= \Phi \otimes I_{\mathcal{X}} (\beta\beta^*).$$

Comparing last 2 lines, $\Phi(M) = \sum_{i=1}^{\mathrm{rank}(J(\Phi))} A_i M A_i^*$.

$\boxed{5} \Rightarrow \boxed{4}$

$\boxed{7} \Rightarrow \boxed{6}$: immediate.

$\boxed{4} \Rightarrow \boxed{6}$: take $A = \sum_{a \in \Gamma} A_a \otimes e_a$, $\mathcal{Z} = \mathbb{C}^{\Gamma}$

$\boxed{5} \Rightarrow \boxed{7}$:

$\boxed{6} \Rightarrow \boxed{1}$: $\Phi_1(M) = AMA^*$ CP.

$\Phi_2(M') = \mathrm{Tr}_{\mathcal{Z}}(M')$ CP.

$\Phi = \Phi_2 \circ \Phi_1 = \mathrm{Tr}_{\mathcal{Z}}(AMA^*)$ CP.

5 Sec 5.3.2 Characterization of trace preserving (TP) linear maps

Def: $\Phi \in T(X, Y)$ is <u>unital</u> if $\Phi(\mathbb{1}_X) = \mathbb{1}_Y$.

<u>Thm 5.4</u> $\forall \Phi \in T(X, Y)$, TFAE

$\boxed{1}$ Φ TP

$\boxed{2}$ Φ^* unital

$\boxed{3}$ $\mathrm{Tr}_Y(J(\Phi)) = \mathbb{1}_X$

$\boxed{4}$ $\exists \Gamma$, $\{A_a : a \in \Gamma\}$, $\{B_a : a \in \Gamma\} \subseteq L(X, Y)$

s.t. $\sum_{a \in \Gamma} A_a^* B_a = \mathbb{1}_X$

$\Phi(M) = \sum_{a \in \Gamma} A_a M B_a^*$

$\boxed{5}$ All Kraus reps satisfy .

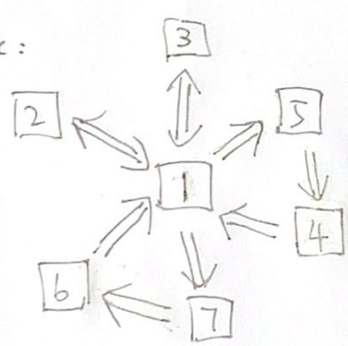
$\boxed{6}$ $\exists A, B \in L(X, Y \otimes Z)$

s.t. $A^* B = \mathbb{1}_X$

$\Phi(M) = \mathrm{tr}_Z(A M B^*)$

$\boxed{7}$ All Stinespring rep satisfy .

Pf structure:



Pf: Lemma: If $N \in L(X)$ satisfies "$\forall M \in L(X)$, $\mathrm{tr}\, NM = \mathrm{tr}\, M$"

then $N = \mathbb{1}_X$.

Pf(Lemma): express $N = \sum_a c_a W_a$ where $\{W_a\}_a$ = set of Weyl operators. $W_0 = \mathbb{1}_X$

To prove $N = \mathbb{1}_X$ by contradiction, suppose $N \neq \mathbb{1}_X$.

$\therefore$ Either (i) $\exists a \neq 0$ s.t $c_a \neq 0$ or $c_a = 0$

or (ii) $\forall a \neq 0$, $c_a = 0$ and $c_0 \neq 1$.

If (i), take $M = W_a^*$. $\mathrm{tr}\, M = 0$

then, $\mathrm{tr}\, M = 0 = \mathrm{tr}\, MN = c_a \dim(X)$

$\therefore c_a = 0$ contradiction.

If (ii) $N = c_0 \mathbb{1}_X$. let $M = \mathbb{1}_X$.

$$\mathrm{tr} M = \dim X$$
$$\|$$
$$\mathrm{tr} NM = \mathrm{tr}\, c_0 \mathbb{1}_X = c_0 \dim X$$

$\therefore c_0 = 1$ contradiction.

Thinking more, the Lemma must be immediate by the def of the dual space.

$\mathrm{Tr}\, N M = \langle N^*, M \rangle = \langle \mathbb{1}_X, M \rangle \quad \forall M \quad \Rightarrow \quad N^* = \mathbb{1}_X$

Pf (Thm 5.4)

(i) We first prove the $\boxed{1} \Rightarrow \boxed{5} \Rightarrow \boxed{4}$ cycle.

Let $\Phi(M) = \sum_{a \in \Gamma} A_a M B_a^*$ be <u>any</u> Kraus map.

$$\left.\begin{aligned} tr\left(\Phi(M)\right) &= \sum_{a\in\Gamma} tr(A_a M B_a^*) && (\text{tr linear}) \\ &= \sum_{a\in\Gamma} tr(B_a^* A_a M) && (\text{tr cyclic}) \\ &= tr\left[\left(\sum_{a\in\Gamma} B_a^* A_a\right) \cdot M\right] && (\text{tr linear}) \end{aligned}\right\} (*)$$

Given $\boxed{1}$, Φ TP $\therefore \forall M \ \ tr(M) = tr(\Phi(M))$.

By lemma, $\sum_{a\in\Gamma} B_a^* A_a = \mathbb{1}_X$.

$\boxed{5} \Rightarrow \boxed{4}$ immediate.

Given $\boxed{4}$, apply $(*)$ to that Kraus rep, replace $\sum_{a\in\Gamma} B_a^* A_a$ by $\mathbb{1}_X$ in the end and get $tr\, M$

$\therefore \Phi$ TP.

(ii) Similarly, for the $\boxed{1} \Rightarrow \boxed{7} \Rightarrow \boxed{6}$ cycle:

(*)

Let $\Phi(M) = \mathrm{tr}_Z (A M B^*)$ be any lin map in Stinespring rep.
$A, B \in L(X, Y \otimes Z)$

$$\mathrm{tr}\, \Phi(M) = \mathrm{tr}_Y (\mathrm{tr}_Z\, A M B^*)$$

$$= \mathrm{tr} (A M B^*)$$

$$= \mathrm{tr} (B^* A\, M)$$

Given $\boxed{1}$, Φ TP $\therefore$ last line $= \mathrm{tr}\, M$

By lemma, $B^* A = \mathbb{1}_X$.

$\boxed{7} \Rightarrow \boxed{6}$ immediate.

Given $\boxed{6}$, apply (*) to the Stinespring map,

replace $B^* A$ by $\mathbb{1}_X$ to get $\mathrm{tr}\, M$

$\therefore \Phi$ TP.

(iii) For $\boxed{1} \Leftrightarrow \boxed{3}$:

Recall $\Phi(M) = \mathrm{tr}_X \left(\mathbb{1}_Y \otimes M^T \right) J(\Phi)$

$$\mathrm{tr}\, \Phi(M) = \underbrace{\mathrm{tr}_Y\ \mathrm{tr}_X}_{= \mathrm{tr} \,=\, \mathrm{tr}_X\, \mathrm{tr}_Y} (\mathbb{1}_Y \otimes M^T) J(\Phi)$$

implicitly on Y

$$= \mathrm{tr} \left[\mathrm{tr}_Y (\mathbb{1}_Y \otimes M^T) J(\Phi) \right]$$

implicitly on X

$$\mathrm{tr}_Y (\mathbb{1}_Y \otimes M^T) J(\Phi) = M^T \cdot \left[\mathrm{tr}_Y\, J(\Phi) \right]$$

$\therefore \Phi$ TP $\Leftrightarrow$ $\forall M$ $\mathrm{tr}\left[M^T \cdot \mathrm{tr}_Y J(\Phi)\right] = \mathrm{tr} M^T = \mathrm{tr} M$ $\overset{\text{lemma}}{\Leftrightarrow}$ $\mathrm{tr}_Y J(\Phi) = \mathbb{1}_X$.

useful !!

Lemma: $\forall A \in L(\mathcal{X}),\ N \in L(\mathcal{Y} \otimes \mathcal{X})$

$$\mathrm{tr}_{\mathcal{Y}} (\mathbb{1}_{\mathcal{Y}} \otimes A)\, N = A \cdot \mathrm{tr}_{\mathcal{Y}} N .$$

Pf: Check on a basis for A & a basis for N.

$A = |a\rangle\langle b|$, $|a\rangle, |b\rangle \in \mathcal{X}$ $\quad$ $N = |c\rangle\langle d| \otimes |a'\rangle\langle b'|$

$|c\rangle, |d\rangle \in \mathcal{Y}$, $|a'\rangle, |b'\rangle \in \mathcal{X}$

(iv) For $\boxed{1} \Leftrightarrow \boxed{2}$:

use $\Phi \in T(\mathcal{X}, \mathcal{Y}) = L(L(\mathcal{X}), L(\mathcal{Y}))$

$\forall A \in L(\mathcal{X}), B \in L(\mathcal{Y})$, $\quad \langle B, \Phi(A) \rangle = \langle \Phi^*(B), A \rangle$

Take $B = \mathbb{1}_{\mathcal{Y}}$, $\quad \langle B, \Phi(A) \rangle = \mathrm{tr}\, \Phi(A)$, $\quad \langle \Phi^*(B), A \rangle = \mathrm{tr}\, (\Phi^*(\mathbb{1}_{\mathcal{Y}}))^* A$

$\boxed{1} \Leftrightarrow \underset{\text{for all } A}{\text{LHS} = \mathrm{tr}\, A} \Leftrightarrow \underset{\text{by lemma}}{(\Phi^*(\mathbb{1}_{\mathcal{Y}}))^* = \mathbb{1}_{\mathcal{X}}} \Leftrightarrow \Phi^*(\mathbb{1}_{\mathcal{Y}}) = \mathbb{1}_{\mathcal{X}}$

$\Updownarrow$

Φ^* unital $\boxed{12}$

⑥ Sec 5.3.3 Characterization of Q channels.

Def $C(\mathcal{X},\mathcal{Y})$ = set of all Q channels from $\mathcal{X}$ to $\mathcal{Y}$.

ie $C(\mathcal{X},\mathcal{Y}) = \{ \Phi \in T(\mathcal{X},\mathcal{Y}) : \Phi \text{ TP \& CP} \}$.

Cor 5.5: Let $\Phi \in T(\mathcal{X},\mathcal{Y})$. TFAE:

$\boxed{1}$ $\Phi \in C(\mathcal{X},\mathcal{Y})$

$\boxed{2}$ $J(\Phi) \in \mathrm{Pos}(\mathcal{Y} \otimes \mathcal{X})$ and $\mathrm{tr}_{\mathcal{Y}}(J(\Phi)) = \mathbb{1}_{\mathcal{X}}$

$\boxed{3}$ $\exists$ finite Γ, $\{A_a : a \in \Gamma\} \subseteq L(\mathcal{X},\mathcal{Y})$

s.t. $\Phi(M) = \sum_{a\in\Gamma} A_a M A_a^*$ and $\sum_{a\in\Gamma} A_a^* A_a = \mathbb{1}_{\mathcal{X}}$

this hold $\forall M \in L(\mathcal{X})$
implicit in defined Φ
as a function of M

$\boxed{4}$ $\boxed{3}$ holds for some Γ with $|\Gamma| = \mathrm{rank}(J(\Phi))$.

$\boxed{5}$ $\exists$ CES $\mathcal{Z}$, isometry $A \in \mathcal{U}(\mathcal{X}, \mathcal{Y}\otimes\mathcal{Z})$

s.t. $\Phi(M) = \mathrm{Tr}_{\mathcal{Z}} AMA^*$

$\boxed{6}$ $\boxed{5}$ holds for $\mathcal{Z} = \mathbb{C}^{\mathrm{rank}(J(\Phi))}$.