Sec 2.4 the "vec" ...

$$\mathrm{vec} : L(\mathcal{X}, \mathcal{Y}) \rightarrow \mathcal{Y} \otimes \mathcal{X}$$

$$\mathrm{vec}(E_{a,b}) = e_b \otimes e_a \quad , \qquad a \in \Sigma_{\mathcal{X}}, \ b \in \Sigma_{\mathcal{Y}}$$

ie $\mathrm{vec}(|b\rangle\langle a|) = |b\rangle |a\rangle$.

① vec is clearly reversible !, a bijection

② vec just relabels the classical states ....

③ $\langle A, B \rangle = \langle \mathrm{vec}(A), \mathrm{vec}(B) \rangle$

To show the above, note both sides linear in A, B

$\therefore$ suffices to check both sides are identical on a basis of $L(\mathcal{X}, \mathcal{Y})$.

$\therefore$ Consider $\boxed{\begin{array}{l} A = |b\rangle\langle a|\, e^{i\theta} \\ B = |d\rangle\langle c|\, e^{i\alpha} \end{array}}$

$$\mathrm{LHS} = \langle A, B \rangle = \mathrm{Tr}(A^* B) = \mathrm{Tr} \overset{e^{-i\theta} e^{i\alpha}}{\wedge} |a\rangle\langle b|d\rangle\langle c| = \overset{e^{-i\theta} e^{i\alpha}}{\wedge} \langle c|a\rangle \langle b|d\rangle .$$

$$\mathrm{RHS} = \langle \underset{e^{i\theta} |b\rangle |a\rangle}{\mathrm{vec}(A)}, \underset{|d\rangle |c\rangle e^{i\alpha}}{\mathrm{vec}(B)} \rangle = \left(\underset{e^{-i\theta}}{\wedge} \langle b| \langle a| \right) \left(\underset{e^{i\alpha}}{\wedge} |d\rangle |c\rangle \right) = \underbrace{\langle a|c\rangle}_{\langle c|a\rangle} \langle b|d\rangle \quad e^{-i\theta} e^{i\alpha}$$

for basis states $|a\rangle, |c\rangle$

④ For general $u \in \mathcal{X}, v \in \mathcal{Y}$

$$\mathrm{vec}(u v^*) = u \otimes \bar{v} \quad , \quad \text{or } \mathrm{vec}(|u\rangle\langle v|) = |u\rangle |\bar{v}\rangle .$$

Pf: let $|u\rangle = \sum_{b \in \Sigma_{\mathcal{Y}}} u(b) |b\rangle$, $|v\rangle = \sum_{a \in \Sigma_{\mathcal{X}}} v(a) |a\rangle$

$$\mathrm{vec}(|u\rangle\langle v|) = \mathrm{vec} \left(\sum_b u(b) |b\rangle \right) \left(\sum_a \overline{v(a)} \langle a| \right)$$

$$= \sum_{ba} u(b) \overline{v(a)} \, \mathrm{vec}(|b\rangle\langle a|) \qquad \text{(linearity)}$$

$$= \sum_{ba} u(b)\,\overline{v(a)}\;|b\rangle|a\rangle \qquad \text{(def of vec)}$$

$$= \left(\sum_b u(b)\,|b\rangle\right)\left(\sum_a \overline{v(a)}\,|a\rangle\right)$$

$$= |u\rangle\langle\bar{v}|$$

5 For any CESs $\mathcal{X}_1, \mathcal{X}_2, \mathcal{Y}_1, \mathcal{Y}_2$

$\forall A \in L(\mathcal{X}_1, \mathcal{Y}_1),\ B \in L(\mathcal{X}_2, \mathcal{Y}_2),\ C \in L(\mathcal{X}_2, \mathcal{X}_1)$

$$(A \otimes B)\,\mathrm{vec}(C) = \mathrm{vec}(A C B^T)$$

6 For any CESs $\mathcal{X}, \mathcal{Y}$, $\forall A, B \in L(\mathcal{X}, \mathcal{Y})$

$$\mathrm{Tr}_{\mathcal{X}}\left(\mathrm{vec}(A)\,\mathrm{vec}(B)^*\right) = AB^*$$

$$\mathrm{Tr}_{\mathcal{Y}}\left(\mathrm{vec}(A)\,\mathrm{vec}(B)^*\right) = (B^*A)^T$$

Pf 5 6 Assignment 1. We have not yet covered $\otimes$ & $\mathrm{Tr}_{\mathcal{X}}, \mathrm{Tr}_{\mathcal{Y}}$ here but they are in QIC710 so you can start A1.

We will cover $\otimes$, $\mathrm{Tr}_{\mathcal{X}}, \mathrm{Tr}_{\mathcal{Y}}$ before end of lecture 4 Sep 19.

A1 due Sep 26.

So, what on earth is vec(A)?

7. Thm: Let $A \in L(\mathcal{X}, \mathcal{Y})$, $\beta = \sum_{a \in \Sigma_\mathcal{X}} e_a \otimes e_a$

Then $(A \otimes \mathbb{1})\beta = \mathrm{vec}(A)$.

Pf: Assignment 1.

But this explains why vec is of any interest !!

vec(A) is the "Choi rep" for the complete positive map

$$\rho \mapsto A \rho A^*$$

These maps are the building blocks of the Kraus maps

$$\rho \mapsto \sum_k A_k \rho A_k^*$$

that are Q channels that we will study in detail lec 3 onwards.

8. Recall the transpose trick $(A \otimes \mathbb{1})\beta = (\mathbb{1} \otimes A^T)\beta$

$$\begin{array}{ccc} (A \otimes \mathbb{1})\beta & = & (\mathbb{1} \otimes A^T)\beta \\ \| & & \| \\ \mathrm{vec}(A) & & \mathrm{SWAP}(A^T \otimes \mathbb{1})\beta \\ & & \| \\ & & \mathrm{SWAP}(\mathrm{vec}(A^T)) \end{array}$$

$\therefore$ $\mathrm{SWAP} \in L(\mathcal{Y} \otimes \mathcal{X}, \mathcal{X} \otimes \mathcal{Y})$

$\mathrm{SWAP}(|b\rangle|a\rangle) = |a\rangle|b\rangle$.

9. From 7, we have a corollary that, given any $u \in \mathcal{Y} \otimes \mathcal{X}$,

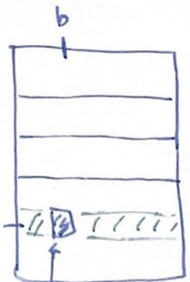
$u = (A \otimes \mathbb{1}_\mathcal{X})\beta$ where $u = \mathrm{vec}(A)$.

From 6 $\mathrm{tr}_\mathcal{X}\, uu^* = AA^* \in L(\mathcal{Y})$.

$\boxed{\text{Thinking } \rho = AA^*, \; u = \text{purification of } \rho}$

↗ use convention on p(7)

(10) $\mathrm{vec}(|a\rangle\langle b|) = |a\rangle|b\rangle$, $|a\rangle \in \mathcal{Y}$, $|b\rangle \in \mathcal{X}$

$A =$ 

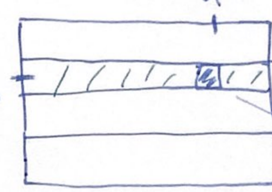
$\uparrow$

$L(\mathcal{X},\mathcal{Y})$

(a, b) entry

$A = |a\rangle\langle b|$

$\xrightarrow{\text{transpose}}$

$A^T =$ 

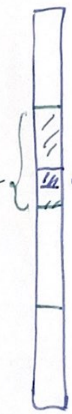
$\uparrow$

$L(\mathcal{Y},\mathcal{X})$

(b, a) entry of A^T

$|b\rangle\langle a|$

$\downarrow$ vec

$|b\rangle|a\rangle =$ 

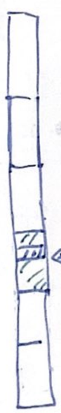
b-th block

a-th entry in the b-th block

$\downarrow$ vec

SWAP

SWAP^{-1}

$|a\rangle|b\rangle =$ 

a-th block → b-th entry in the a-th block.

$\uparrow$

$\mathcal{Y} \otimes \mathcal{X}$

" range over b"

the a-th row of A

$\downarrow$ vec

the a-th block of vec(A).

"stacking the columns".

"stacking the rows"