Lecture $\overline{\underline{V}}$

The Threshold Theorem:

Definitions & level reduction

In the next two lectures we will prove the threshold theorem, one of the most important results in quantum information theory.

Fault-tolerant EC (formal def) 2

Def: FT EC Recovery Property

(ECRP) 2 in Lecture 1 p. 13

$\boxed{EC}$ satisfies ECRP if

whenever $s \leq t$ $\quad t = \lfloor \frac{d-1}{2} \rfloor$

$$\equiv \boxed{EC} \equiv \quad = \quad \equiv \boxed{EC} \equiv \boxed{f_s} \equiv$$

↑ s faults

where $\boxed{f_s}$ is an <u>s-filter</u>

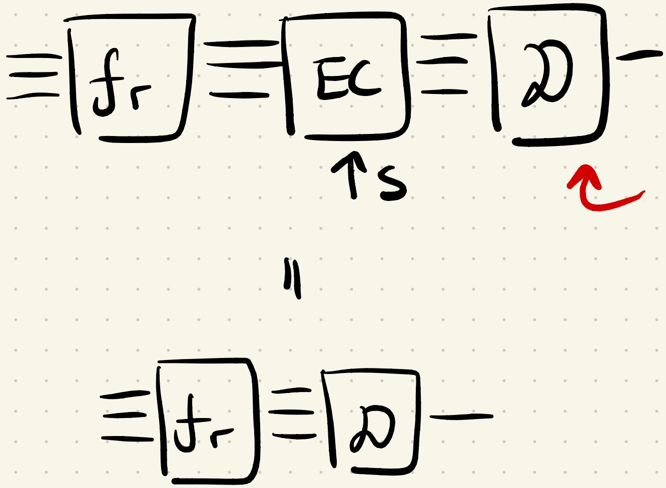
ie a projector onto the subspace spanned by codewords w/ $\leq s$ errors

Def: FT EC Correctness Property (ECCP)

① in lecture 1 p.12

$\boxed{EC}$ satisfies ECCP if

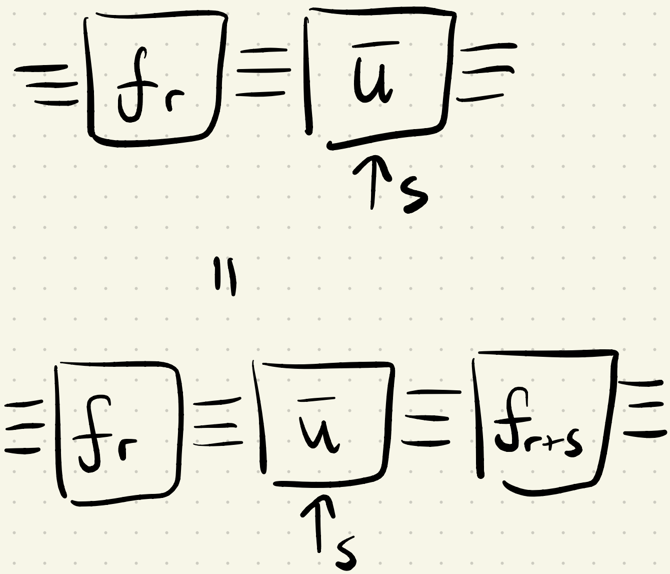
whenever $r+s \leq t$



ideal decoder

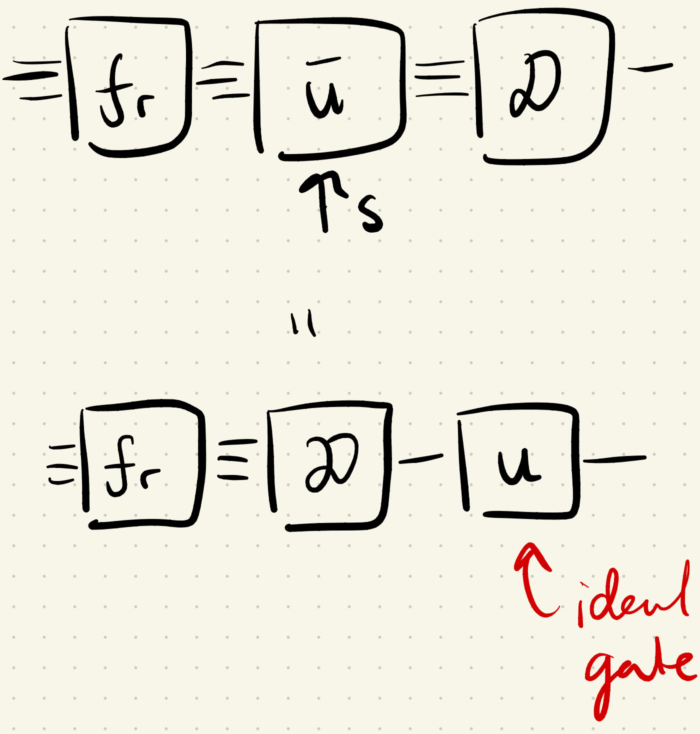
Def: FT Gate Error Propagation Property (GPP)

1q logical gate $\boxed{\bar{u}}$ satisfies GPP if, whenever $r+s \leq t$,



Def: FT Gate Correctness Property (GCP)

A 1q logical gate $\boxed{\bar{u}}$ satisfies GCP if, whenever $r+s \leq t$



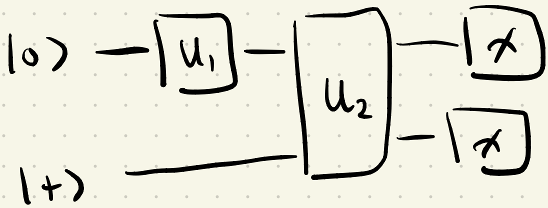
Can check these conditions for the FT EC & gate implementations we covered in prev. lectures.

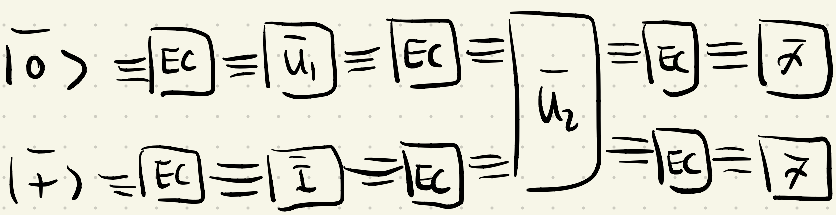
(Also analogous defs for state prep. & meas.)

We assume ECRP, ECCP, GPP, GCP hold.

Extended rectangles (exRecs)

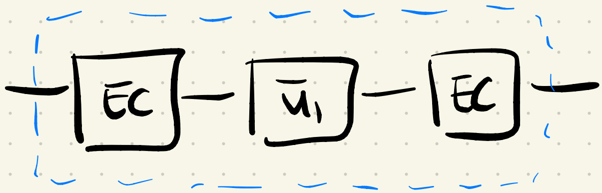
Recall the fault-tolerant version of a circuit



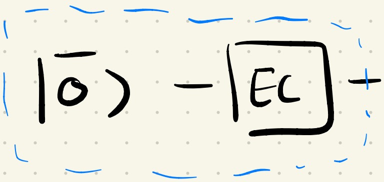


(Encoding in $k=1$ QECC)

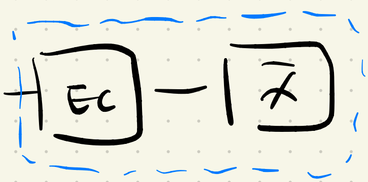
An exRec consists of



Logical gate + EC before & after



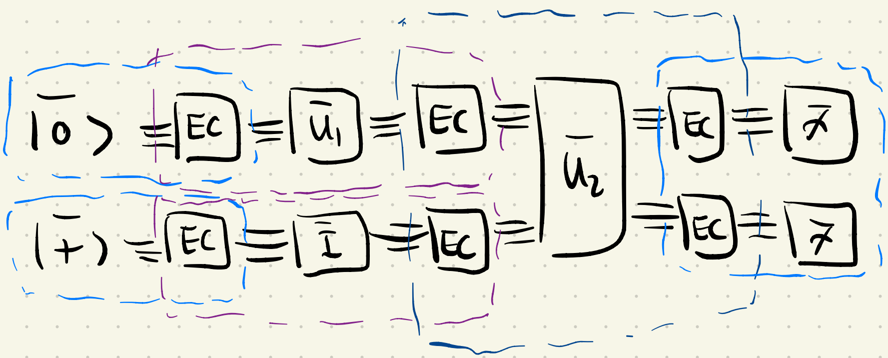
Logical state prep + EC after



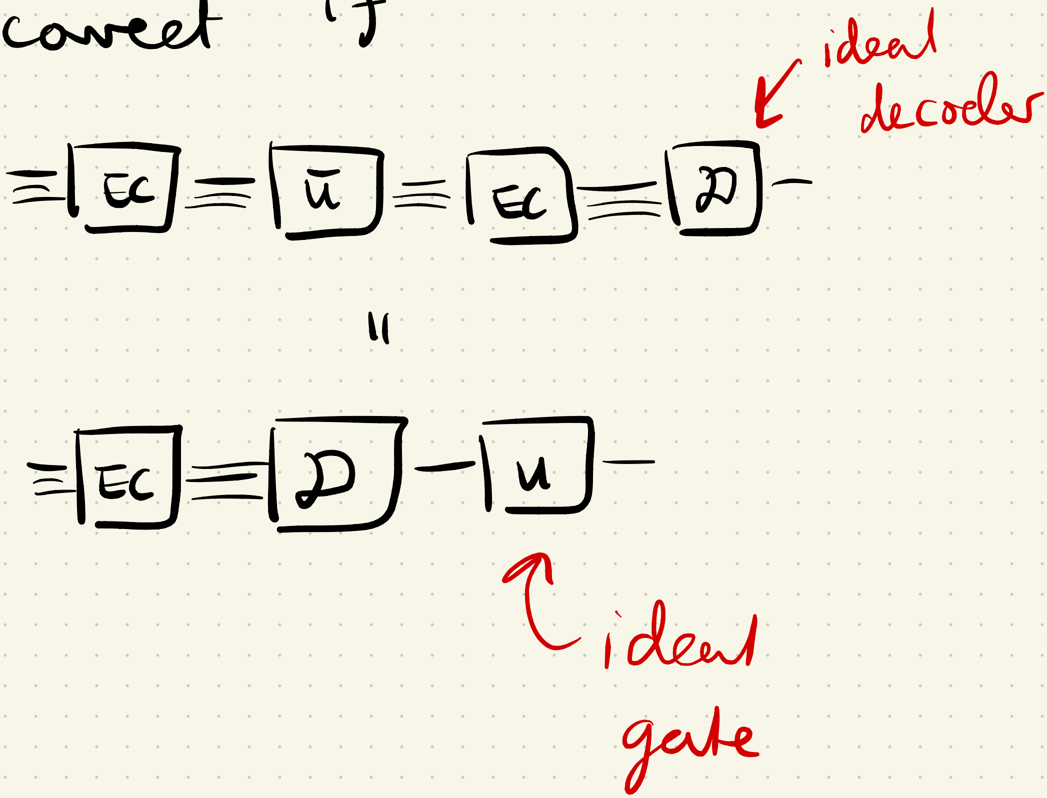
Logical measurement + EC before

If the underlying QECC corrects t errors, an exRec is good if it contains no more than t faults and is bad otherwise.

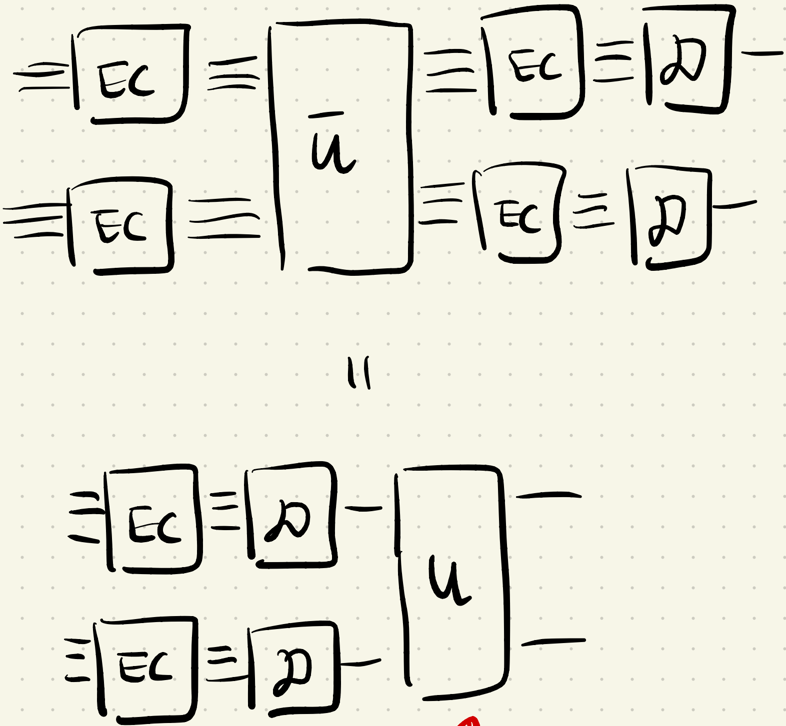
exRecs overlap



A 1q gate exRec is correct if

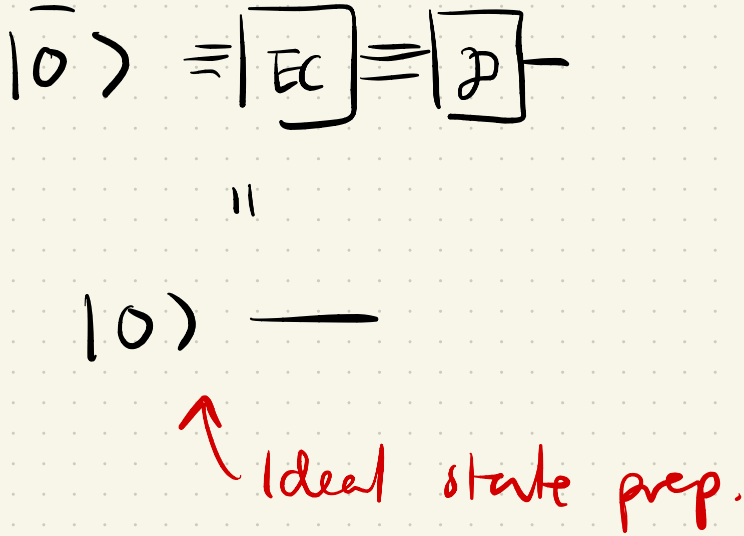


A 2q gate exRec is correct if

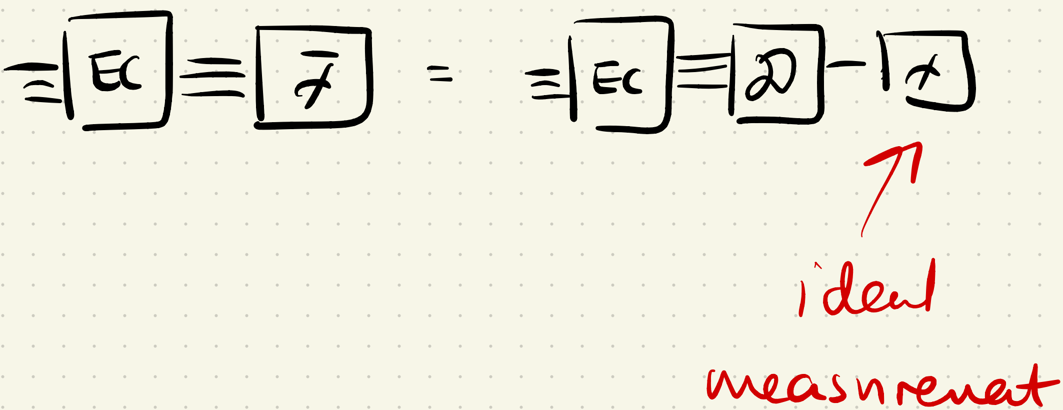


Ideal gate

A state prep. exRec is correct if

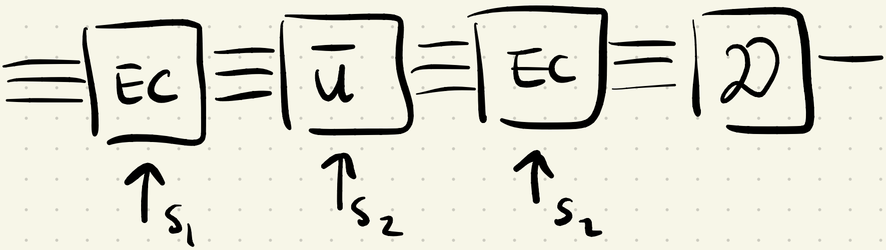


A measurement exRec is correct if



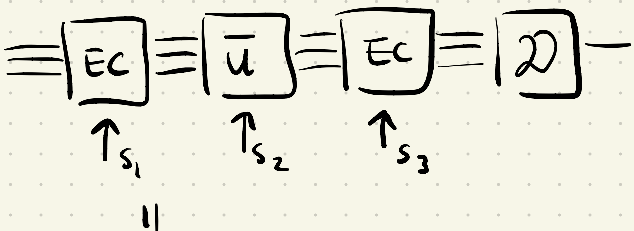
Thm: A good exRec is a correct exRec

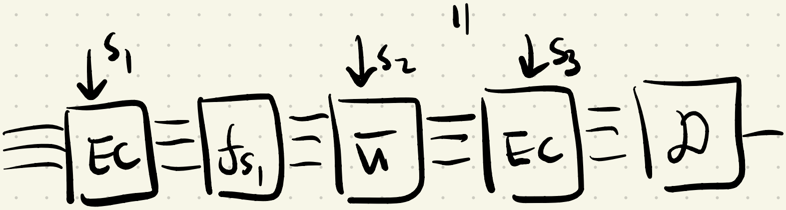
Proof (1q gates)



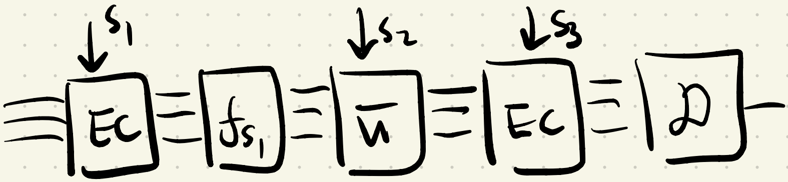
$s_1+s_2+s_3 \le t$ (Good exRec)

Use ECRP





Use GPP



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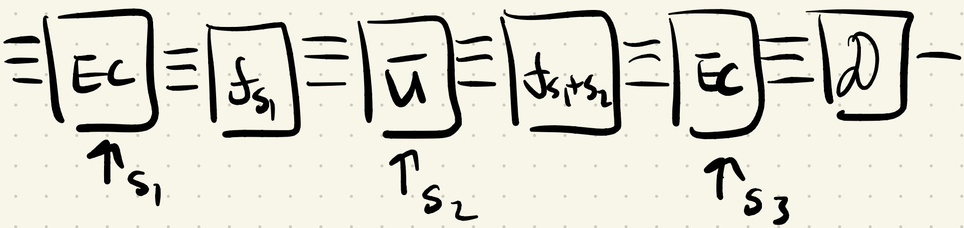
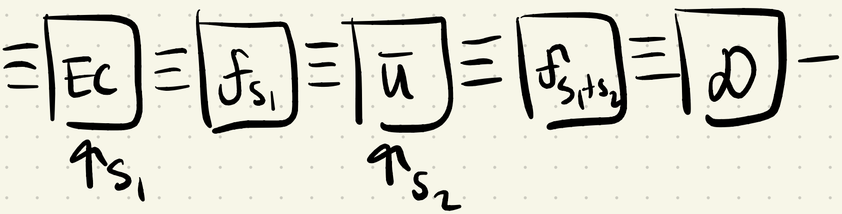
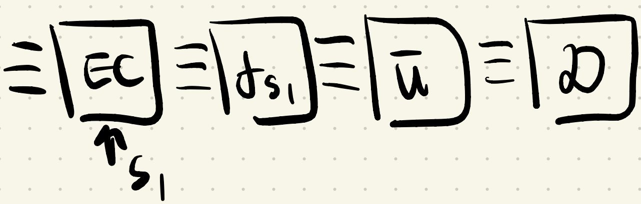


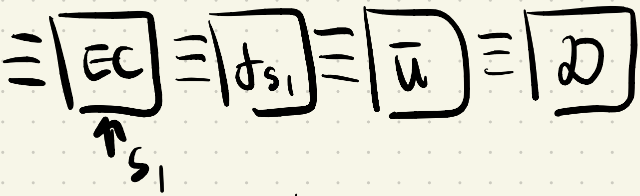
= (ECCP)



= (reverse GPP)



Use GCP



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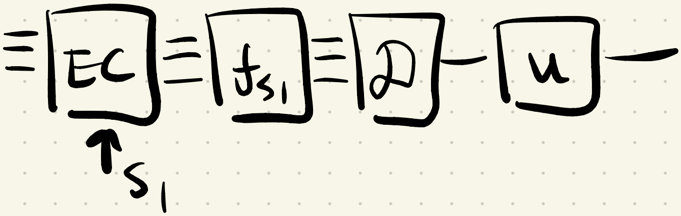
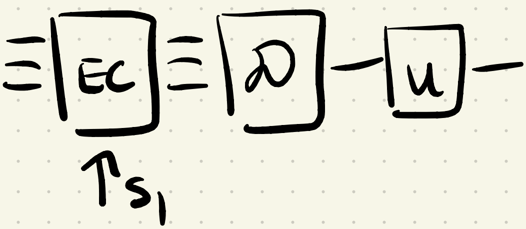


= (reverse ECRP)

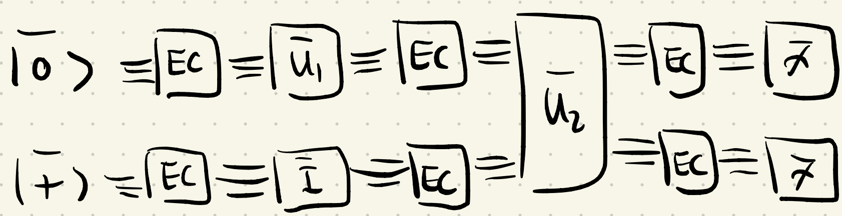


□

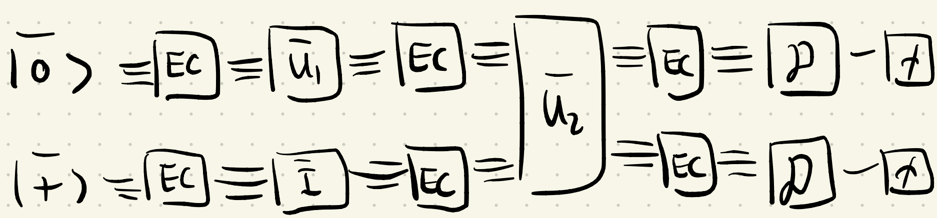
Proof for other location types is similar

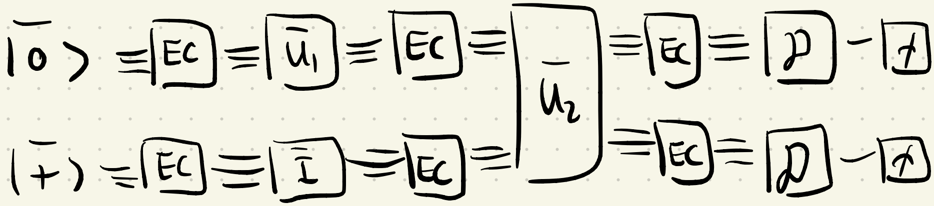
Corr: If all exRecs in an FT circuit are good then the output distribution is the same as the output distribution of the ideal circuit.

Proof sketch:

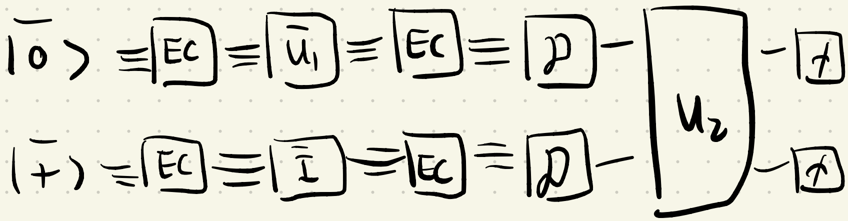


= (Correct meas. exRecs)

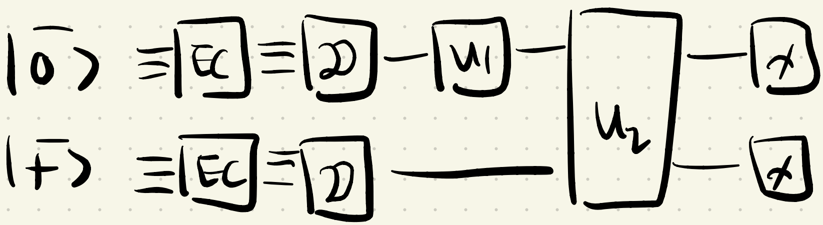




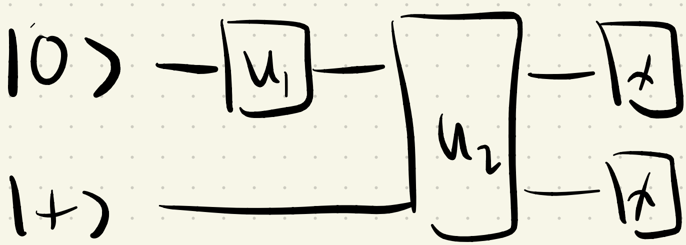
= (Correct 2q exRec)



= (Correct 1q exRecs)



= (Correct state prep. exRecs)



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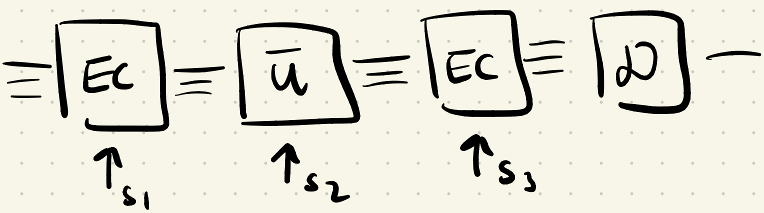
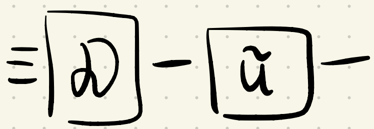
But what if an exRec is bad?

We want to replace bad exRecs with faulty locations e.g.



$(s_1+s_2+s_3 > t)$

=



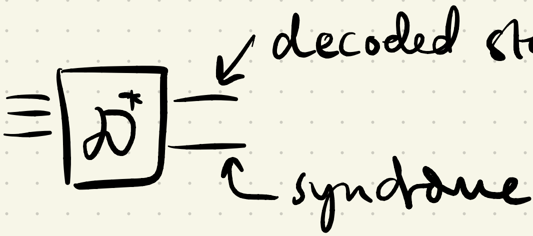
noisy gate

But the noisy gate $\tilde{U}$ will in general depend on the noise in the exRec and the error syndrome of the state entering the exRec.

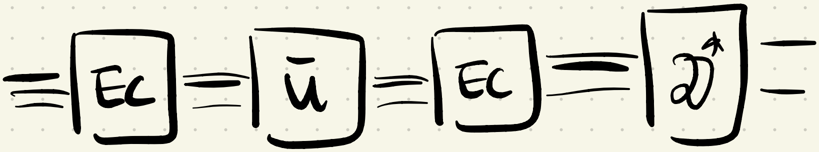
The solution is to keep track of the error syndrome information.

Def : $*$ - decoder $\boxed{\mathcal{D}^*}$

works just like the ideal decoder but keeps the syndrome information



One can show



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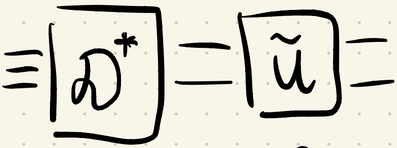

noisy gate that depends on syndrome

$*$ - decoder is unitary U_*

Let E = 

$$E U_* = \underbrace{U_* U_*^\dagger E U_*}_{\text{noisy gate } \tilde{U}}$$

Recap

We have shown that we can replace good exRecs by ideal locations and bad exRecs by faulty locations.

The obvious next question is: how many bad exRecs do we expect?

To answer this question we must first choose a noise model.

<u>Def</u>: an error model is local stochastic if for any set of faults R, $\Pr[S \subseteq R]$

$$= \sum_{R | S \subseteq R} \Pr[R] \leq p^{|S|}$$

where $0 < p < 1$.

This is more general than iid noise and can include adversarial noise.

Assuming a local stochastic noise model, what is the prob. that a single exRec is bad?

Suppose the exRec contains A locations.

exRec is bad if it contains $> t$ faults.

Then

union bound

$$\Pr[\text{exRec bad}] \leq \sum_{|S|=t+1} \sum_{R|S \subseteq R} \Pr[R]$$

$$= \binom{A}{t+1} p^{t+1}$$

includes sets R of size $t+2$ etc.
(In fact we overcount these.)

We can now prove our main theorem, the level reduction theorem.

Thm: Suppose we have a fault tolerant circuit simulating an unencoded circuit C, subjected to a local stochastic noise model w/ error prob. p.

Then the FT circuit is equivalent to C subjected to a local stochastic noise model w/ error prob. p' where

$$p' \leq \binom{A}{t+1} p^{t+1}$$

and A is the number of locations in the largest exRec in the FT circuit.

Proof :

For any given run of the FT circuit there will be some set of faults R that occur w/ prob. $\Pr[R]$

For R we assign good and bad exRecs.

Then the FT circuit is equivalent to C w/ faults at the locations corresponding to bad exRecs.

We need to show:

given a set of r exRecs in the FT circuit the probability that there exists a set of faults R such that each of the r exRecs is bad is at most p'^r.

The set of exRecs is bad if every exRec has t+1 or more faults.

There are at most

$\binom{A}{t+1}^r$ sets of locations

with $t+1$ locations in each

of the r exRecs.

Each such set has a

total of $(t+1)r$ locations.

$\Rightarrow$ Prob. of a set of

faults containing all these

locations is $\leq p^{(t+1)r}$

Union bound

$\Rightarrow$ Prob. of $t+1$ faults

on each of the r exRecs is

$$\binom{A}{t+1}^r p^{(t+1)r} = \left[\binom{A}{t+1} p^{t+1}\right]^r$$

ie local stochastic noise

w/ error prob. $p' = \binom{A}{t+1} p^{t+1}$

$\square$.

[There is a subtlety about overlapping exRecs that I have neglected, see Gottesman notes]