QIC 820 / CO781 / CO486 / CS 867 F2023 A3

Q1. For the Quantum State Discrimination problem consider the ensemble $\{(p_i, \rho_i)\}_{i=1,\dots,n}$

where $p_i = \frac{1}{n} \quad \forall i = 1,\dots,n$

$\rho_i \in D(X)$, ρ_i pure $\forall i = 1,\dots,n$

and $\frac{1}{n} \sum_{i=1}^{n} \rho_i = \frac{\mathbb{1}_X}{\dim(X)}$

ⓐ Using complementary slackness, show that the measurement

$$M_K = \rho_K \cdot \frac{\dim(X)}{n}, \qquad K = 1,\dots,n$$

is optimal.

ⓑ What is the optimal prob of correctly determining what ρ_K is given?

NB You can use all the results shown in class.

2 marks ⓐ

1 mark ⓑ.

Q2. Fix $H \in \mathrm{Herm}(\mathcal{Y} \otimes \mathcal{X})$.

Consider the optimization

channels from $\mathcal{X}$ to $\mathcal{Y}$.

$$\sup \{ \langle H, J(\Psi) \rangle : \Psi \in \mathrm{C}(\mathcal{X}, \mathcal{Y}) \}$$

Prove: (a) The sup can be attained by some $\Phi \in \mathrm{C}(\mathcal{X}, \mathcal{Y})$

(b) Φ optimal $\Leftrightarrow$ $\mathrm{Tr}_{\mathcal{Y}}(H\, J(\Phi)) \in \mathrm{Herm}(\mathcal{X})$

and $\mathbb{1}_{\mathcal{Y}} \otimes \mathrm{Tr}_{\mathcal{Y}}(H J(\Phi)) \geq H$

(Prove $[\Rightarrow]$, $[\Leftarrow]$ separately.)

1 mark (a)
3 marks (b) $\Rightarrow$
1 mark (b) $\Leftarrow$

Q3. Unambiguous state discrimination

Bob is given $\rho_i \in D(X)$ with prob p_i.

When asked "What is i?", he can say "I don't know" or must give the correct answer.

Goal: max prob of giving the correct answer

This can be formulated as:

$$\max \sum_i p_i \langle M_i, \rho_i \rangle$$

s.t. $M_1 + M_2 + \dots + M_n \leq \mathbb{1}_X$ (can add $\underset{\geq 0}{M_{n+1}}$ s.t. $\sum_{i=1}^{n+1} M_i = \mathbb{1}_X$) corr to ans "I don't know".

$M_i \in \mathrm{Pos}(X) \quad \forall i = 1, \dots, n$

$\forall i = 1, 2, \dots, n \quad \langle M_i, \sum_{k \neq i} \rho_k \rangle = 0$ (make-no-mistake constraint)

Sum over $k = 1, 2, \dots, i-1, i+1, \dots n$

or equivalently:

$\alpha = \max \langle A, X \rangle$ where $A = \sum_{i=1}^{n} |i\rangle\langle i| \otimes p_i \rho_i$

s.t. ① $\mathrm{tr}_{\mathbb{C}^{n+1}} X = \mathbb{1}_X$

② $\langle X, C_i \rangle = 0$ where $C_i = |i\rangle\langle i| \otimes \sum_{k \neq i} \rho_k$

for each $i = 1, 2, \dots, n$

③ $X \in \mathrm{Pos}(\mathbb{C}^{n+1} \otimes X)$

Note $(\langle i| \otimes \mathbb{1}_X) X (|i\rangle \otimes \mathbb{1}_X)$ is M_i.

(a) 4 marks
(b) 1 mark
(c) 4 marks
(d) 3 marks

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(a) Show that the dual is

$$\beta = \inf \ \mathrm{tr}\, Y_0$$

s.t. $\mathbb{1}_{\mathbb{C}^{n+1}} \otimes Y_0 + \sum_{i=1}^{n} y_i C_i \geq A$

$Y_0 \in \mathrm{Herm}(X)$, $\forall i = 1, \dots, n$, $y_i \in \mathbb{R}$.

(End of Oct 17 lecture will be useful to handle multiple linear constraints.)

(b) Show that $\alpha = \beta$ and α can be attained.

(c) If each ρ_i is pure, $\rho_i = |a_i\rangle\langle a_i|$

and each $|a_i\rangle$ is a linear combination of the other $|a_j\rangle$'s for $j \neq i$

show that unambiguous state discrimination is impossible.

(ie Prob(I don't know) = 1)

(Hint: show that $Y_0 = 0$ with appropriate $y_1, \dots, y_n$ is dual feasible. You need to simplify the dual in (a) a little.)

(d) let $n = 3$, $|a_1\rangle = |0\rangle^{\otimes 2} \in \mathbb{C}^2 \otimes \mathbb{C}^2$, $p_1 = p_2 = p_3 = \frac{1}{3}$

$$|a_2\rangle = \left(\frac{|0\rangle}{2} + \frac{\sqrt{3}}{2}|1\rangle\right)^{\otimes 2}$$

$$|a_3\rangle = \left(\frac{|0\rangle}{2} - \frac{\sqrt{3}}{2}|1\rangle\right)^{\otimes 2}$$

Note $\alpha = 0.75$: each $|a_i\rangle$ in symmetric subspace with 3 dims,

no need to show

M_1 also in " "

and orthogonal to $|a_2\rangle, |a_3\rangle$, etc. Can optimize directly.

Find feasible Y to get good upper bound on β.

(NB: Primal SDP is not strictly feasible $\therefore$ complementary slackness doesn't hold.)

Make sure upper bound < 1, and see if you can get close to 0.75 (OK if not).