QIC 820 / CO781 / CO786 / CS 867 F2023 A2

Q1 Q supermap / super-super operator

Intro So $A \in L(X, Y)$, A takes $v \in X$ to $Av \in Y$

$\Phi \in T(X, Y)$, Φ takes $A \in L(X)$ to $\Phi(A) \in L(Y)$

$\quad \parallel$

$L(L(X), L(Y))$

$\Phi \in C(X, Y)$ iff $\underset{\wedge}{\overset{\forall Z}{\Phi \otimes I_Z}}$ takes $\rho \in D(X \otimes Z)$ to $\Phi \otimes I(\rho) \in D(Y \otimes Z)$

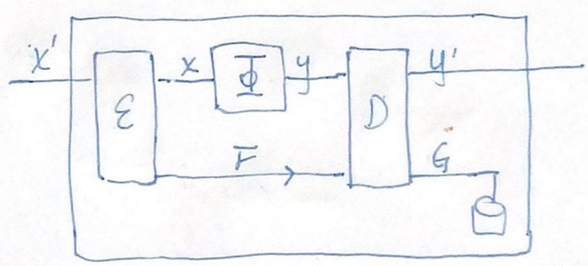
iff Φ is TP & CP.

i.e. what lin transformations take channels to channels?

$\left\{ \begin{array}{l} \text{Here we want to study } S \in L(\,T(X, Y),\ T(X', Y')\,) \\ \text{and what properties ensure } S \in SM(\,C(X, Y),\ C(X', Y')\,). \end{array} \right.$

The end goal is to show $S \in SM(\,C(X, Y),\ C(X', Y')\,)$

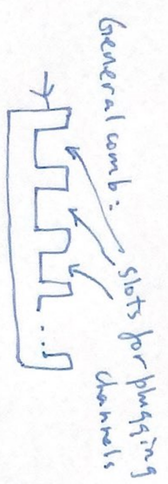
$\Leftrightarrow \forall \Phi \in C(X, Y)$, $S(\Phi) =$

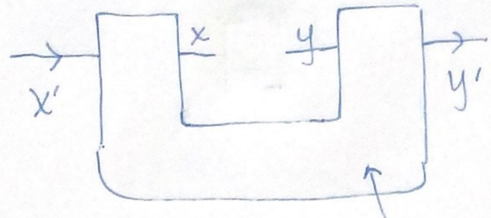


for some $\mathcal{E}$, D, F, G independent of Φ.

Two interpretations: ① S is a forward-assisted code, turning any input channel Φ to a new channel $S(\Phi)$ with encoder $\mathcal{E}$, forward communication F, and decoder D.

② $\mathcal{S}$ is a Q comb:

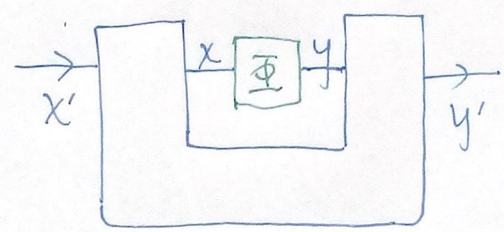




made of circuit with input X', Y and output X, Y'

We can plug any $\Phi \in \mathcal{C}(X, Y)$ in the slot and the composition is $\mathcal{S}(\Phi)$

$\mathcal{S}(\Phi) =$



The assignment is taken from EPL, 83 (2008) 30004 by Chiribella, D'Ariano, Perinotti.

The question is broken into small steps in red boxes, so you can try without consulting the paper. You can use results from previous steps in later parts.

You can use the paper, since the proofs aren't exactly easy.

Watch out for the typos and overlap notations that are not consistent with the assignment writing.

But you must write your own solutions, and in the notations developed in the course consistent with the assignment writing and justify your work based on course materials.

Goal: to digest what we learnt about channels, partial trace, and their representations.

(a) super-complete-positivity.

Let $S \in SM(\, C(X,Y),\, C(X',Y'))$ be arbitrary

Let $I \in SM(\, C(K,H),\, C(K,H))$ be the identity supermap.

ie $\forall\, \Psi \in C(K,H)$, $I(\Psi) = \Psi$.

(*) Suppose $S \otimes I \in SM(\, C(X \otimes K, Y \otimes H),\, C(X' \otimes K, Y' \otimes H))$

ie if Φ is an arbitrary Q channel from $X \otimes K$ to $Y \otimes H$
then $S \otimes I(\Phi)$ is a Q channel from $X' \otimes K$ to $Y' \otimes H$.

Let $\mathcal{S} \in L(\, L(Y \otimes X),\, L(Y' \otimes X'))$ be the map induced by S

on choi matrices.

ie $\forall\, \Phi$, $\mathcal{S}(\, J(\Phi)) = J(\, S(\Phi))$

Explain why :

(*) $\Rightarrow \forall E$ $\quad \mathcal{S}(E) = \sum_i S_i E S_i^*$ $\quad$ for some $S_i \in L(\, Y \otimes X,\, Y' \otimes X')$

in $L(Y \otimes X)$

3 marks

(b) Super-trace-preserving step 1

6 marks

Suppose $C \in \mathrm{Pos}(\mathcal{Y} \otimes \mathcal{X})$ satisfies the following condition.

$$\forall \Phi \in C(\mathcal{X}, \mathcal{Y}), \quad \mathrm{Tr}(C \cdot J(\Phi)) = 1$$

Show that $C = \mathbb{1}_{\mathcal{Y}} \otimes \rho$ for some $\rho \in D(\mathcal{X})$.

Hint:

① It suffices to show $\forall E \in \mathrm{Pos}(\mathcal{Y} \otimes \mathcal{X})$. $\mathrm{tr}(CE) = \mathrm{tr}((\mathbb{1}_{\mathcal{Y}} \otimes \rho) E)$.

② Pick E s.t. $\mathrm{tr}_{\mathcal{Y}} E \leq \mathbb{1}_{\mathcal{X}}$, $\mathrm{tr}_{\mathcal{Y}} E \neq \mathbb{1}_{\mathcal{X}}$.

Then, E is the Choi matrix of a CP map that is not TP.

Now, you want to add D = Choi matrix of another CP map

s.t. $D+E$ is the Choi matrix of a channel (TP)

③ try $D = \sigma \otimes (\mathbb{1} - \mathrm{tr}_{\mathcal{Y}} E)$, and note $\sigma \otimes \mathbb{1}$ = Choi matrix of some channel!

Note eventually, you can show that $\rho = \mathrm{tr}_{\mathcal{Y}}\left(C(\sigma \otimes \mathbb{1})\right)$

- Note that conversely $\forall \rho \in D(\mathcal{X})\ \forall \Phi$ $\mathrm{tr}(\mathbb{1}_{\mathcal{Y}} \otimes \rho)\, J(\Phi) = \mathrm{tr}(\Phi(\rho^T)) = 1$.

(c) Super-trace-preserving part 2.

Let $\mathcal{S}^*$ be the dual of $\mathcal{S}$ defined in part (c).

Recall that $\mathcal{S}^*(M) = \sum_i S_i^* E S_i$.
$\wedge$
$\forall M \in L(Y' \otimes X')$

$$\boxed{\text{Show that } (*) \Rightarrow \forall \xi \in D(X'),\ \mathcal{S}^*(\mathbb{1}_{Y'} \otimes \xi) = \mathbb{1}_Y \otimes N(\xi) \quad \wedge \quad \exists N \in C(X', X) \text{ s.t.}}$$

Hint: show that the following choice of C

$$C = \mathcal{S}^*(\mathbb{1}_{Y'} \otimes \xi)$$

satisfy the condition of C in part (b), and apply (b).

$\boxed{\text{Relate } \rho \text{ in the conclusion of part (b) to } N \text{ and } \xi.}$

(d) super-trace-preserving part 3.

$$\boxed{\text{Show that } (*) \Rightarrow \forall E \in L(Y \otimes X),\ \mathrm{tr}_{Y'} \mathcal{S}(E) = N^*(\mathrm{tr}_Y E)}$$

Hint: show $\forall \xi \in D(X')$, $\mathrm{Tr}(\xi \cdot \mathrm{tr}_{Y'} \mathcal{S}(E)) = \mathrm{tr}(\xi \cdot N^*(\mathrm{tr}_Y E))$

tr over X'

Hint: show that this equals to

$$\mathrm{Tr}_{X'Y'}((\mathbb{1}_{Y'} \otimes \xi) \cdot \mathcal{S}(E))$$

Note that part (d) shows that indeed S takes trace-preserving Φ to trace-preserving $S(\Phi)$.

To see this, recall $\mathcal{E}$ TP $\Leftrightarrow$ $\mathrm{tr}_{out}\, J(\mathcal{E}) = \mathbb{1}_{in}$.

Now, $\mathrm{tr}_{y'}\, \mathcal{S}(J(\Phi))$

$= N^*(\mathrm{tr}_y\, J(\Phi))$ due to part (d)

$= N^*(\mathbb{1}_x)$ If Φ is TP

$= \mathbb{1}_{x'}$

so $S(\Phi)$ is also TP.

No surprise! We assumed this in (*)
but the axiomatic assumption in (*) is now translated to concrete structural properties of S and $\mathcal{S}$!

(e) Structural theorem for $\mathcal{S}$ or $\mathcal{S}$.

From (d), $\forall E \in L(\mathcal{Y} \otimes \mathcal{X})$, $\mathrm{tr}_{\mathcal{Y}'} \mathcal{S}(E) = \mathcal{N}^*(\mathrm{tr}_{\mathcal{Y}} E)$

If the channel $\mathcal{N} \in C(\mathcal{X}', \mathcal{X})$ has Kraus rep

$$\forall M' \in L(\mathcal{X}'), \quad \mathcal{N}(M') = \sum_l N_l M' N_l^* \qquad \left(N_l \in L(\mathcal{X}', \mathcal{X})\right)$$

then the dual has Kraus rep

$$\forall M \in L(\mathcal{X}), \quad \mathcal{N}^*(M) = \sum_l N_l^* M N_l \qquad \left(N_l^* \in L(\mathcal{X}, \mathcal{X}')\right)$$

Remember $\mathcal{N}^*$ is lin, CP, unital (not necessarily TP).

Using the Kraus rep for $\mathcal{S}$ in part (a)

and .... $\mathcal{N}^*$ above, we have

$$\mathrm{tr}_{\mathcal{Y}'} \sum_i S_i E S_i^* = \sum_l N_l^* (\mathrm{tr}_{\mathcal{Y}} E) N_l$$

Take $\{|k'\rangle\}$ to be o.n basis for $\mathcal{Y}'$

$\{|k\rangle\}$ .... $\mathcal{Y}$.

$$\sum_i \sum_{k'} \left(\langle k'| \otimes \mathbb{1}_{\mathcal{X}'}\right) S_i E S_i^* \left(|k'\rangle \otimes \mathbb{1}_{\mathcal{X}}\right)$$

$$= \sum_l \sum_k N_l^* \left(\langle k| \otimes \mathbb{1}_{\mathcal{X}}\right) E \left(|k\rangle \otimes \mathbb{1}_{\mathcal{X}}\right) N_l$$

$$= \sum_l \sum_k \left(\langle k| \otimes N_l^*\right) E \left(|k\rangle \otimes N_l\right)$$

Sanity check: both sides in $L(\mathcal{X}')$.

Both sides are Kraus reps of a CP lin map from $L(\mathcal{Y} \otimes \mathcal{X})$ to $L(\mathcal{X}')$.

By a theorem in Nielsen & Chuang and text book by Prof Watrous (likely in QIC 710, LN 2011) the Kraus operators on the two sides are related.

The Kraus rep with fewer Kraus op is the canonical one.

So, by choosing $N(M') = \sum_l N_l M' N_l^*$ to be canonical (with fewest # of terms) and by choosing $\{|k\rangle\}$ to be o.n (thus min #) the botton line is canonical.

(#) $\forall$ isometory W s.t.

$$(\langle k'| \otimes \mathbb{1}_{\mathcal{X}'}) S_i = \sum_{k,l} W_{k'i,\,kl} \left(\langle k| \otimes N_l^*\right)$$

with $W^\dagger W = \mathbb{1}$.

If $i \in \Sigma_1$, let $\mathcal{A} = \mathbb{C}^{\Sigma_1}$, if $l \in \Sigma_2$, let $\mathcal{B} = \mathbb{C}^{\Sigma_2}$.

then $W_{k'i,\,kl}$ defines a matrix W taking $|k\rangle|l\rangle$ to $|k'\rangle|i\rangle$ (see P8 .... Part 1-Lec1.pdf), i.e. $W \in L(\mathcal{Y} \otimes \mathcal{B}, \mathcal{Y}' \otimes \mathcal{A})$.

($) $\{ \; W_{k'i,\,kl} = \langle k'|\langle i| W |k\rangle|l\rangle$

Sub ($) into (#):

$$(\langle k'|\otimes \mathbb{1}_{X'})\, S_i = \sum_{k,l}\Big(\langle k'|\langle i|\, W\, |k\rangle|l\rangle\Big)\Big(\langle k|\otimes N_l^{*}\Big)$$

* Show that the above implies

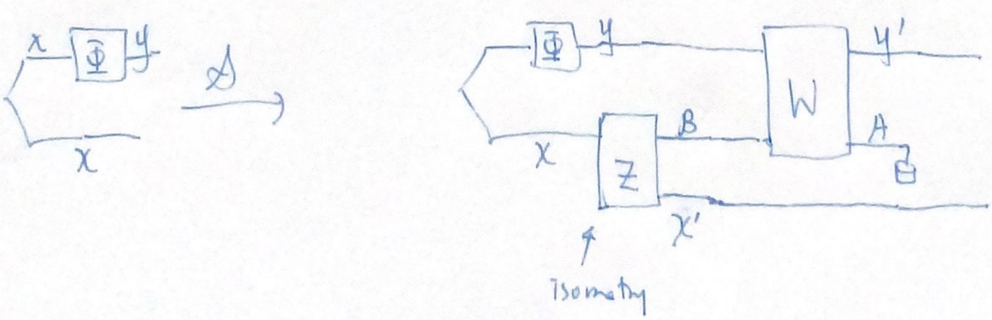
$$\underset{L(Y'\otimes X',\, Y\otimes X)}{S_i} = \Big[\big(\mathbb{1}_{Y'}\otimes\langle i|_A\big)\underset{L(Y\otimes B,\, Y'\otimes A)}{W}\Big]\otimes \mathbb{1}_{X'} \overset{\text{matrix multi}}{\cdot} \Big(\mathbb{1}_{Y}\otimes \overbrace{\sum_l \underset{L(\mathbb{C},\, B)}{|l\rangle}\otimes \underset{L(X,\, X')}{N_l^{*}}}^{Z}\Big)$$

So sub above to kraus rep of $\mathcal{A}$:

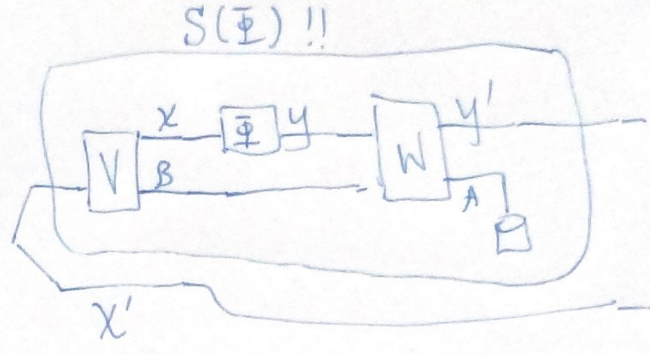
$$\mathcal{A}(E) = \sum_i \Big[\big(\mathbb{1}_{Y'}\otimes\langle i|_A\big)W\Big]\otimes\mathbb{1}_{X'}\cdot(\mathbb{1}_Y\otimes Z)\cdot E\cdot(\mathbb{1}_Y\otimes Z^{*})\cdot\Big[W^{*}\big(\mathbb{1}_{Y'}\otimes|i\rangle_A\big)\otimes\mathbb{1}_X\Big]$$

$$= \mathrm{tr}_A\,(W\otimes\mathbb{1}_{X'})\cdot(\mathbb{1}_Y\otimes Z)\cdot E\cdot(\mathbb{1}_Y\otimes Z^{*})(W^{*}\otimes\mathbb{1}_X)$$

Let $J(\Phi)=E$.



$$= $$



$V = \sum_{l} |l\rangle \otimes \bar{N}_l$ = partial transpose of Z.

Credit: Prof John Watrous F2019 offering

Q2 Let $\rho_0, \rho_1 \in \mathrm{D}(\mathcal{X})$ be density operators, for some complex Euclidean space $\mathcal{X}$, let n be a positive integer, and define real numbers $\alpha, \beta \in [0,1]$ as

$$\alpha = \frac{1}{2}\|\rho_0 - \rho_1\|_1 \quad \text{and} \quad \beta = \mathrm{F}(\rho_0, \rho_1).$$

5 marks (a) Prove that

$$1 - \exp\left(-\frac{n\alpha^2}{2}\right) \leq \frac{1}{2}\left\|\rho_0^{\otimes n} - \rho_1^{\otimes n}\right\|_1 \leq n\alpha.$$

4 marks (b) For density operators $\sigma_0, \sigma_1 \in \mathrm{D}(\mathcal{X}^{\otimes n})$ defined as

$$\sigma_0 = \frac{1}{2^{n-1}} \sum_{\substack{a_1,\ldots,a_n \in \{0,1\} \\ a_1+\cdots+a_n \text{ even}}} \rho_{a_1} \otimes \cdots \otimes \rho_{a_n},$$

$$\sigma_1 = \frac{1}{2^{n-1}} \sum_{\substack{a_1,\ldots,a_n \in \{0,1\} \\ a_1+\cdots+a_n \text{ odd}}} \rho_{a_1} \otimes \cdots \otimes \rho_{a_n},$$

prove that

$$1 - n\beta \leq \frac{1}{2}\|\sigma_0 - \sigma_1\|_1 \leq \exp\left(-\frac{n\beta^2}{2}\right).$$

When answering both parts of this question, it may be helpful to make use of the fact that for every real number $\lambda \geq 1$, the inequality

$$\left(1 - \frac{1}{\lambda}\right)^{\lambda} < \frac{1}{e}$$

is satisfied.