QIC 820 / CS 867 / CO781 A1

Covering lectures 1-4, Sept 7-19, due Sep 19 5pm on Crowdmark

Q1. Let $\mathcal{X}, \mathcal{X}' = \mathbb{C}^{\Sigma_X}$, $\mathcal{Y} = \mathbb{C}^{\Sigma_Y}$ be CESs,

$$\beta = \sum_{a \in \Sigma_X} e_a \otimes e_a \in \mathcal{X} \otimes \mathcal{X}',$$

$$A, B \in L(\mathcal{X}, \mathcal{Y}).$$

(a) Prove that $(A \otimes \mathbb{1}_X)\,\beta = \text{vec}(A)$

(b) Prove that $\text{Tr}_X\left(\text{vec}(A)\,\text{vec}(B)^*\right) = AB^*$

$$\text{Tr}_Y\left(\text{vec}(A)\,\text{vec}(B)^*\right) = (B^* A)^T$$

(c) For any CESs $\mathcal{X}_1, \mathcal{Y}_1, \mathcal{X}_2, \mathcal{Y}_2$, show that

$\forall\, A \in L(\mathcal{X}_1, \mathcal{Y}_1)$, $B \in L(\mathcal{X}_2, \mathcal{Y}_2)$, $C \in L(\mathcal{X}_2, \mathcal{X}_1)$

$$(A \otimes B)\,\text{vec}(C) = \text{vec}(ACB^T).$$

Q2 (a) Let $\mathcal{X}, \mathcal{X}', \beta$ be as defined in Q1.

Suppose registers XX' are initially in the joint state $\frac{1}{\dim(\mathcal{X})}\beta\beta^*$.

Suppose the meas $\mu : \Gamma \to \text{Pos}(\mathcal{X})$ is applied to X.
(destructive)

Show that the post-meas state is $\frac{1}{\dim(\mathcal{X})} \sum_{i \in \Gamma} e_i e_i^* \otimes \mu(i)^T$

where $\{e_i\}$ is a standard o.n basis for $\mathbb{C}^\Gamma$.

Q2 (b) Let $\phi : \mathrm{Herm}(X) \to \mathbb{R}^\Gamma$ be a linear function.

Suppose $\forall \rho \in D(X)$, $\phi(\rho) \in P(\Gamma)$ (set of all prob vectors over Γ)

Find a measurement $\mu : \Gamma \to \mathrm{Pos}(X)$

s.t. $\phi(\rho) = \sum_{i \in \Gamma} \langle \mu(i), \rho \rangle e_i e_i^*$. (‡)

Please present $\mu(i)$ in terms of ϕ, and show that

$\forall i \in \Gamma$, $\mu(i) \in \mathrm{Pos}(X)$, $\sum_{i \in \Gamma} \mu(i) = \mathbb{1}_X$,

before verifying eq (‡) above.

Q3. Programmable gate-array. each $X_i \cong X$

The proof for teleportation implies the following.

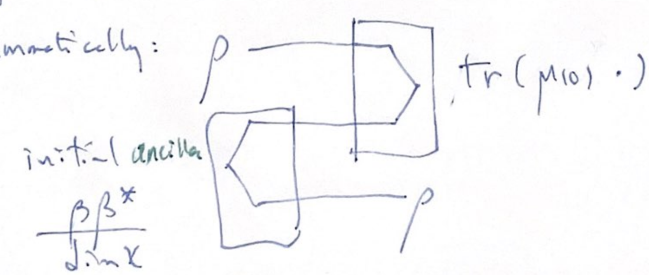
For registers $X_1 X_2 X_3$, initially in the state $\rho \otimes \frac{\beta\beta^*}{\dim(X)}$,

applying a measurement μ with

$$\mu(0) = \frac{\beta\beta^*}{\dim X}$$

and getting the outcome 0 puts the state ρ in X_3.

Diagrammatically:



(a) Show that if (i) $\mu(0)$ is replaced by $U \otimes \mathbb{1} \cdot \mu(0)\, U^* \otimes \mathbb{1}$

(ii) initial ancilla is replaced by $I \otimes \Phi \left(\frac{\beta\beta^*}{\dim X} \right)$

any Quantum channel

X_3 will be in the state $\Phi(U^* \rho U)$.

Q3 (b) Let $\sum_{k \in \Gamma} p_k U_k \tau U_k^\dagger = \frac{\mathbb{1}_X}{d} = \Delta(\tau)$ be the depolarizing channel on X.

Show that if $\mu(k) = p_k \, U_k \otimes \mathbb{1} \, \mu(0) \, U_k^* \otimes \mathbb{1}$

then $\{\mu(k)\}_{k \in \Gamma}$ defines a meas on $X_1 X_2$.

Q3 (c) Explain how to apply to ρ a random U_k followed by Φ, if you have the initial ancilla

$I \otimes \Phi \left(\frac{\beta \beta^*}{\dim X} \right)$.

Explain also why you cannot apply a specific U_k followed by Φ.

(hint: you time travel...)

Q1 (a) 2 marks
(b) 2
(c) 2
} routine

Q2 (a) 3 marks ← routine
(b) 5 marks ← nontrivial.

Q3 (a) 2 marks
(b) 2 mark
(c) 2 mark
} question is intentionally mathematically less precise
this is an exercise to convert vague ideas to precise statements.