Question 1. The 5-qubit QECC [6 marks]
Recall that the 5-qubit QECC has four generators for the stabilizer group:
\[
\begin{align*}
G_1 &= X \otimes Z \otimes Z \otimes X \otimes I \\
G_2 &= I \otimes X \otimes Z \otimes Z \otimes X \\
G_3 &= X \otimes I \otimes X \otimes Z \otimes Z \\
G_4 &= Z \otimes X \otimes I \otimes X \otimes Z
\end{align*}
\]
(a)[3 marks] List the 16 possible 0- or 1-qubit Pauli errors for this code. For each of these errors, write down the ± outcome resulting from measuring each of the 4 generators. You can provide the answers in a table, similar to the one we have started in class.
(b)[3 marks] Show that \( H^\otimes 5 \) is not a logical operation for this code.

Question 2. Encoded \( R \) gate on 7-qubit code [7 marks]
We define the \( R \) gate as the \( 2 \times 2 \) unitary (up to a phase) satisfying the following commutation relations:
\[
RXR^\dagger = iXZ, \quad RZR^\dagger = Z.
\]
Side remark: if we consider \( Z \) as a \( \pi/2 \) rotation, \( T \) as a \( \pi/8 \) rotation, \( R \) is a \( \pi/4 \) rotation, all along the \( z \)-axis, and up to a phase, \( R = \sqrt{Z} = T^2 \). \( R \) is in the Clifford group. We choose to specify \( R \) using commutation relation and not bother with the irrelevant overall phase.
Recall from class that the 7-qubit Steane code has stabilizer group generated by
\[
\begin{align*}
G_1 &= I \otimes I \otimes I \otimes X \otimes X \otimes X \\
G_2 &= I \otimes X \otimes X \otimes I \otimes I \otimes X \otimes X \\
G_3 &= X \otimes I \otimes X \otimes I \otimes X \otimes I \otimes X \\
G_4 &= I \otimes I \otimes I \otimes Z \otimes Z \otimes Z \\
G_5 &= I \otimes Z \otimes Z \otimes I \otimes I \otimes Z \otimes Z \\
G_6 &= Z \otimes I \otimes Z \otimes I \otimes Z \otimes I \otimes Z
\end{align*}
\]
with \( X_L = X^\otimes 7 \), \( Z_L = Z^\otimes 7 \). In this question you will show that \( U = R^\otimes 7 Z^\otimes 7 \) effects a transversal, encoded, \( R \) gate on the 7-qubit code.
(a)[4 marks] For each \( G_i \), \( i = 1, \cdots, 6 \), write down \( UG_iU^\dagger \) as a product of the above generators (thus \( U \) is an encoded operation on the 7-bit code). Because of the symmetry in \( U \) and similarities in the generators, it suffices to show your work/reasoning for \( UG_1U^\dagger \) and \( UG_4U^\dagger \), and state the answers for the rest.
(b)[3 marks] Show that \( UX_LU^\dagger = iX_LZ_L \) and \( UZ_LU^\dagger = Z_L \) (thus showing \( U \) is an encoded \( R \) gate).
**Question 3. A magic multi-purpose 4-qubit code** [13 marks + 4 marks bonus]

Consider a stabilizer code $C$ whose stabilizer group $S$ is generated by

$$G_1 = X \otimes X \otimes X \otimes X$$
$$G_2 = Z \otimes Z \otimes Z \otimes Z$$

$C$ encodes 2 qubits into 4 qubits.

(a) [3 marks] Explain why the following 4 matrices are encoded operations. Explain why we can choose them as the encoded Pauli $X$ and $Z$ operators on the two encoded qubits.

$$X_{1L} = X \otimes X \otimes I \otimes I$$
$$Z_{1L} = I \otimes Z \otimes Z \otimes I$$
$$X_{2L} = I \otimes X \otimes X \otimes I$$
$$Z_{2L} = I \otimes I \otimes Z \otimes Z$$

(b) [2 marks] Show that $H \otimes H \otimes H \otimes H$ is an encoded operation.

(c) [4 marks] (bonus) What encoded operation does $H^\otimes 4$ perform? (Hint: check commutation relation with the encoded $X$ and $Z$'s, and recall that each element of the stabilizer group is an encoded identity operator.)

(d) [2 marks] Find the codewords $|00_L\rangle, |01_L\rangle, |10_L\rangle, |11_L\rangle$ using the stabilizer generators and the encoded Pauli operators $X_{1L}, Z_{1L}, X_{2L}, Z_{2L}$ given in part (a).

(e) Show that an erasure on any of the 4 qubits can be corrected. By symmetry, it suffices to show that erasure on the first qubit can be corrected.

(i) [2 marks] Suppose one of $I, X, Y, Z$ happens to the first qubit. What are the outcomes if $G_1$ and $G_2$ are measured?

(ii) [2 marks] State a method to correct the erasure on the first qubit. Detail explanation is not needed.

(f) [2 marks] Instead of correcting an erasure error, the same code $C$ can be used to detect a single unknown Pauli error. Explain how. (Here, you want to show that there are measurements that distinguish the no error case from the case with any single-qubit Pauli error.)

(g) [0 marks] Demoting a question to a remark:

Consider a new QECC $C'$ obtained by adding $Z_{2L}$ to the list of stabilizer generator. This encodes 1 qubit in 4, and correct 1 amplitude damping error without satisfying the QECC condition!

**Question 4. QECC for the 50-50 erasure channel?** [4 marks]

Consider the quantum operation $\mathcal{E}$ which erases a qubit input with probability 50%. It is called the 50-50 erasure channel. We described it in class, and it has an alternative description:

\[
\begin{array}{c}
A \\
|0\rangle \\
B_2 \\
\frac{1}{\sqrt{2}}(|00\rangle + |11\rangle) \\
E_2
\end{array}
\]

\[
\begin{array}{c}
B_1 \\
E_1 \\
\end{array}
\]

- $A$ is the input qubit.
- $E_1$ and $E_2$ are the outputs of the erasure channel.
- $B_1$ and $B_2$ are the failure indicators.
In the above, $A$ is the input qubit system and $B_1B_2$ are the output systems. $E_1B_2E_2$ are qubit systems initialized in a fixed pure state $|a_{E_1B_2E_2}⟩ = |0⟩_{E_1} \otimes \frac{1}{\sqrt{2}}(|00⟩ + |11⟩)_{B_2E_2}$. Then, a unitary $U$ is applied, which conditioned on $B_2$ being in the state $|1⟩$, swaps $B_1$ and $E_1$. Finally, performing a partial trace of $E_1E_2$ gives the output in systems $B_1B_2$. The information whether a system is erased or not can be found in $B_2$.

In other words, $\mathcal{E}(\rho) = \text{tr}_{E_1E_2} U (\rho_{A} \otimes |a⟩⟨a|_{E_1B_2E_2}) U^\dagger$.

Let $n$ be any positive integer. Explain why there is no QECC that encodes 1 qubit into $n$ qubits such that the encoded qubit can be recovered with very high probability after the noise process $\mathcal{E} \otimes n$. (Hint: you can use the fact that we cannot clone a qubit with very high probability, and find a contradiction if such a QECC exists.)

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**Question 5. Deriving a remote CNOT using 1-bit teleportation** [0 marks]

PRACTICE QUESTION, DO NOT TURN IN.

Suppose two remote parties Alice and Bob wish to perform a CNOT on two qubits in systems $AB$, where the control qubit $A$ is held by Alice, and the target qubit $B$ is held by Bob. We will derive a method for them to do so by using one maximally entangled state $\frac{1}{\sqrt{2}}(|00⟩ + |11⟩)$, one classical bit of communication from Alice to Bob, and one classical bit of communication from Bob to Alice.

First consider the following circuit. A vertical *double line* connecting a measurement box (in the computational basis) to a unitary $U$ means that “conditioned on the measurement outcome being 1, perform $U$, otherwise do nothing”.

(a) [2 marks] Explain why the above circuit performs a CNOT on the two incoming qubits in systems $AB$, and leaves the output in systems $CD$.

(b) [4 marks] Show that the following circuit implements the same transformation as the previous circuit.
(e) [4 marks] Extract from the circuit in part (b) a protocol for Alice and Bob to apply a CNOT on $AB$ using one maximally entangled state $\frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)$, one classical bit of communication from Alice to Bob, and one classical bit of communication from Bob to Alice.