Quantum Error Correction and Fault Tolerance, Winter 2022

Part II: Fault Tolerance

Problem Set 2

Due: Friday March 4 2022 10pm.

1. State injection circuits [5 marks]

The control-S gate is in the third level of the Clifford hierarchy and is diagonal in the Z basis. Recall that $S = \mathrm{diag}(1, i)$ and that control-S, $CS = \mathrm{diag}(1, 1, 1, i)$. We can derive a state injection circuit for CS starting from the standard teleportation circuit shown in Figure 1.

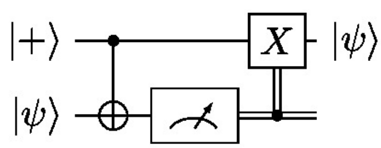

Figure 1: Teleportation circuit

(a) Show that $CS^\dagger(X \otimes X)CS = XS \otimes S^\dagger X$. **[2 marks]**

(b) Commute the CS gate in Figure 2 backwards through the circuit until the input ancilla state is the magic state $CS|++\rangle$ and write down the resultant state injection circuit. **[3 marks]**

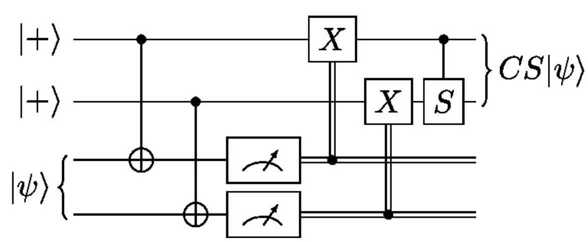

Figure 2: Teleportation circuit with CS

2. Transversal gates of the 5-qubit code [8 marks]

Recall that the stabilizer of the 5-qubit code is generated by the cyclic permutations of $XZZXI$. The logical operators are $\overline{X} = X^{\otimes 5}$ and $\overline{Z} = Z^{\otimes 5}$

(a) Show that $Y^{\otimes 5}$ is an implementation of logical Y in the 5-qubit code. **[1 mark]**

(b) Compute the transformation of X, Y and Z under conjugation by the Clifford gate $K = \exp(\frac{i\pi}{3\sqrt{3}}(X+Y+Z))$. **[4 marks]**

Hint: use the identity $\exp(i\theta\vec{v}\cdot\vec{\sigma}) = \cos\theta I + \sin\theta\vec{v}\cdot\vec{\sigma}$, where $\vec{\sigma} = (X, Y, Z)^T$ and $||\vec{v}|| = 1$.

(c) Show that $\overline{K} = K^{\otimes 5}$ in the 5-qubit code, i.e., this gate is transversal. **[3 marks]**

3. Transversal gates of the [[8,3,2]] code [14 marks]

The [[8,3,2]] code is the smallest known stabilizer code with a transversal non-Clifford gate. It's stabilizer group is

$\langle X^{\otimes 8}, Z_1Z_2Z_3Z_4, Z_5Z_6Z_7Z_8, Z_1Z_2Z_5Z_6, Z_2Z_4Z_6Z_8\rangle$.

An easy way to understand the code is to consider a cube with qubits placed at the vertices, as shown in Figure 3. In this picture, the Z stabilizer generators are associated with faces, i.e., for each face f we have the stabilizer $Z(f) = \prod_{v\in f} Z_v$, where Z_v denotes Z applied to the qubit at vertex v. In this picture we can also define a convenient basis for the logical operators where the logical X operators are associated with faces and the logical Z operators are associated with edges. That is, $\overline{X}_1 = X_1X_2X_3X_4$, $\overline{X}_2 = X_1X_2X_5X_6$, $\overline{X}_3 = X_2X_4X_6X_8$, $\overline{Z}_1 = Z_1Z_5$, $\overline{Z}_2 = Z_1Z_3$ and $\overline{Z}_3 = Z_1Z_2$.

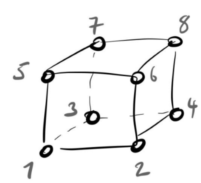

Figure 3: [[8,3,2]] code qubit labels

(a) Write down the encoded computational basis states of the [[8,3,2]] code. **[4 marks]**

(b) Show that $S_2S_4^\dagger S_6^\dagger S_8$ implements a logical $\overline{CZ}_{12}$ gate (acts as CZ on encoded qubits 1 and 2). **[4 marks]**

Hint: you can either apply the gate to the encoded computational basis states or conjugate the logical Pauli operators. Recall that $CZ|xy\rangle = (-1)^{xy}|xy\rangle$ for $x, y \in \{0,1\}$ and $CZ(X\otimes I)CZ^\dagger = X\otimes Z$.

(c) Write down transversal implementations of $\overline{CZ}_{13}$ and $\overline{CZ}_{23}$. **[2 marks]**

(d) Show that $T_1T_2^\dagger T_3^\dagger T_4T_5^\dagger T_6T_7T_8^\dagger$ implements a logical $\overline{CCZ}$ gate. **[4 marks]**

Hint: Recall that $CCZ|xyz\rangle = (-1)^{xyz}|xyz\rangle$ $x, y, z \in \{0,1\}$ and $CCZ(X\otimes I\otimes I)CCZ^\dagger = X\otimes CZ$.