Fall 2020 QIC 890 / CO 781 Assignment 3

Due Nov 20, 2020

Question 1. Accessible information [8 marks]

This question is based on lecture 15, 2020-11-03. Please refer to slides topic-4-3.pdf, p7–14.

There we define the ensemble $\mathcal{E}_2 = \{p(x), \rho_x\}_{x=0}^2$,
where $p(x) = 1/3$ for $x = 0, 1, 2$, $\rho_x = |\psi_x\rangle\langle\psi_x|^{\otimes 2}$,
$|\psi_0\rangle = |0\rangle$, $|\psi_1\rangle = \cos(\pi/3)|0\rangle + \sin(\pi/3)|1\rangle$, $|\psi_2\rangle = \cos(\pi/3)|0\rangle - \sin(\pi/3)|1\rangle$.

Calculate the mutual information between X and the measurement outcome Y for the two measurements $\mathcal{M}_2$ and $\mathcal{M}_3$ defined on p8–10 and p12, and verify that they are ≈ 1.2304 and 1.3695 bits respectively.

Question 2. Some simple exercises on Q boxes [6 marks]

Throughout this question, states are in $\mathbb{C}^2$, and when you are asked to calculate the capacities, **derive the optimal distribution as well**. Hint: $C(Q) = 1$ for (a) and ≈ 0.558 for (b).

(a) Let I, σ_x, σ_y, σ_z denote the Pauli matrices.

$|\psi_1\rangle\langle\psi_1| = \frac{1}{2}(I + \frac{1}{\sqrt{3}}(\sigma_x + \sigma_y + \sigma_z))$

$|\psi_2\rangle\langle\psi_2| = \frac{1}{2}(I + \frac{1}{\sqrt{3}}(\sigma_x - \sigma_y - \sigma_z))$

$|\psi_3\rangle\langle\psi_3| = \frac{1}{2}(I + \frac{1}{\sqrt{3}}(-\sigma_x + \sigma_y - \sigma_z))$

$|\psi_4\rangle\langle\psi_4| = \frac{1}{2}(I + \frac{1}{\sqrt{3}}(-\sigma_x - \sigma_y + \sigma_z))$

Note that $|\langle\psi_i|\psi_j\rangle|$ is constant for $i \neq j$, and the Bloch vectors of these states form the vertices of a tetrahedron.

(i) What is the pretty good measurement corresponding to these states?

(ii) What is the classical capacity of a Q-box capable of emitting $|\psi_i\rangle$ $(i = 1, 2, 3, 4)$?

(b) Consider the states $\rho_0 = |0\rangle\langle 0|$, $\rho_1 = \frac{1}{2}(\frac{I}{2} + |1\rangle\langle 1|)$.

(i) What is the pretty good measurement corresponding to these states?

(ii) What is the classical capacity of a Q-box capable of emitting ρ_0 and ρ_1?

Question 3. The optimal ensemble for $\chi(\mathcal{N})$ [10 marks total, 3 for (a), 7 for (b)]

In this question, you will show that for any quantum channel $\mathcal{N}$ with input and output dimensions d_{in} and d_{out}, the Holevo information of the channel can be optimized by a pure state ensemble of at most d^2 states, where $d = \min(d_{\text{in}}, d_{\text{out}})$. Recall

$$\chi(\mathcal{N}) = \max_{\{p_x,\rho_x\}} \left[S\left(\mathcal{N}\left(\sum_x p_x\rho_x\right)\right) - \sum_x p_x S(\mathcal{N}(\rho_x)) \right]$$

where $\mathcal{E} = \{p_x, \rho_x\}$ is *any* ensemble of states on the input space for $\mathcal{N}$. Our first step is to reexpress the above optimized expression as

$$\chi(\mathcal{N}) = \max_{\rho} \left[S(\mathcal{N}(\rho)) - \min_{\substack{\{p_x, \rho_x\}: \\ \sum_x p_x \rho_x = \rho}} \sum_x p_x S\left(\mathcal{N}(\rho_x)\right) \right] \tag{1}$$

In other words, we first choose the average state ρ for the ensemble, then, we minimize over all possible ensembles $\mathcal{E}_\rho = \{p_x, \rho_x\}$ with average state ρ.

Assume that for any ρ, the minimization of the second term in (1) can be attained by some ensemble $\mathcal{E}_\rho$ with finitely many states (generally mixed), but we have no bound on the number. (This assumption is not needed, but we allow this for the assignment for simplicity.)

(a) Show that for any optimal $\mathcal{E}_\rho = \{p_x, \rho_x\}$ attaining the minimum for the second term in (1) there is a pure state ensemble $\tilde{\mathcal{E}}_\rho = \{q_t, |\psi_t\rangle\langle\psi_t|\}$ with the same average state ρ (thus preserving the first term in (1)) and attaining the same minimum value for the second term in (1).

The Caratheodory's theorem says that, for any set T in $\mathbb{R}^n$, if $a \in \text{conv}(T)$, then, a can be written as a convex combination of at most $n+1$ elements of T.

(b) Let $|\psi_t\rangle$'s be as defined in part (a). Show that there is a distribution r_t with
(1) $r_t > 0$ for at most d^2 values of t,
(2) $\bar{\mathcal{E}} = \{r_t, |\psi_t\rangle\langle\psi_t|\}$ has the same value as the optimal $\mathcal{E}_\rho$ in the first term of (1),
(3) $\bar{\mathcal{E}}$ attains the same minimum in the second term in (1).

Hint for one particular proof method: use Caratheodory's theorem, and recall $d = \min(d_{\text{in}}, d_{\text{out}})$ – it is simpler to first obtain a proof for $d = d_{\text{out}}$ and just explain the difference for the case for $d = d_{\text{in}}$. You can use a proof by contradiction, and strengthen it with the technique of the minimal counterexample.