Limitations on quantum communication

F. Leditzky, Dec. 2020

[Horodecki^3 '98] PPT states are undistillable

A bipartite state ρ_{AB} is called

·) PPT (positive partial transpose), if $\rho_{AB}^{T_B} \geq 0$.

·) distillable, if $\rho_{AB}^{\otimes n}$ can be converted using LOCC into copies of a maximally entangled state at a positive rate (asymptotically faithfully in the limit $n \to \infty$).

Quantum channels:

A channel $\mathcal{N}: A' \to B$ is called PPT if it can only produce PPT states: $(\mathrm{id}_A \otimes \mathcal{N})(\rho_{AA'})$ is PPT for all $\rho_{AA'} \geq 0$.
If it produces bound-entangled states, it is called Horodecki.

·) $\mathcal{N}$ PPT $\iff$ Choi state $\tau_{AB}^{\mathcal{N}}$ is PPT

·) $\mathcal{N}$ Horodecki $\iff$ Choi state $\tau_{AB}^{\mathcal{N}}$ is bound-entangled

·) $\mathcal{N}$ PPT $\Rightarrow$ all output states undistillable $\Rightarrow$ $Q(\mathcal{N}) = 0$

$\boxed{\text{IDEA}}$ Make this quantitative and turn it into general upper bound on $Q(\mathcal{N})$ valid for any $\mathcal{N}$.

Recall: trace norm $\|X\|_1 = \mathrm{tr}\sqrt{X^\dagger X}$

·) ρ_{AB} PPT $\Rightarrow$ $\rho_{AB}^{T_B}$ has all non-negative EV's $\Rightarrow \|\rho_{AB}^{T_B}\|_1 = 1$

·) ρ_{AB} NPT $\Rightarrow$ $\rho_{AB}^{T_B}$ has a negative EV $\Rightarrow \|\rho_{AB}^{T_B}\|_1 > 1$

Superoperators: diamond norm

$$\|\mathcal{N}\|_\diamond = \sup\left\{ \|(\mathrm{id}_A \otimes \mathcal{N})(X_{AA'})\|_1 : \|X_{AA'}\|_1 = 1 \right\}$$

·) $\mathcal{N}$ channel: $\|\mathcal{N}\|_\diamond = 1$

·) $\mathcal{N}$ PPT: $\vartheta \circ \mathcal{N}$ is still a channel $\Rightarrow \|\vartheta \circ \mathcal{N}\|_\diamond = 1$
($\vartheta : X \mapsto X^T$ transpose map)

·) If $\mathcal{N}$ can produce NPT states: $\|\vartheta \circ \mathcal{N}\|_\diamond > 1$.

$\boxed{\text{THM}}$ For any quantum channel $\mathcal{N}$, $Q(\mathcal{N}) \leq \log \|\vartheta \circ \mathcal{N}\|_\diamond$
[Holevo, Werner '01]

Proof ingredients: ·) $\|\vartheta\|_\diamond = \dim \mathcal{H}$

·) $\vartheta \circ \mathcal{N} \circ \vartheta \in$ CPTP iff $\mathcal{N} \in$ CPTP

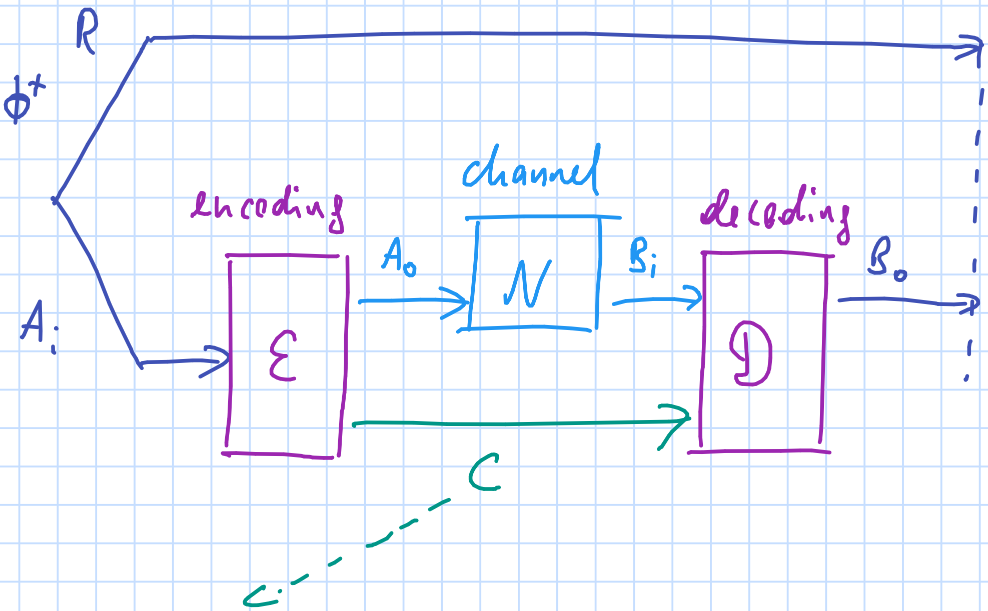
·) $\|\mathcal{N} \otimes \mathcal{M}\|_\diamond = \|\mathcal{N}\|_\diamond \|\mathcal{M}\|_\diamond$ "$\square$"

·) Recovers $Q(\mathcal{N}) = 0$ for PPT-channels. [Watrous '09, '12]

·) Efficiently computable using semidefinite programming.

Upper bound from PPT-assistance

Sending quantum information $\Leftrightarrow$ transmitting entanglement

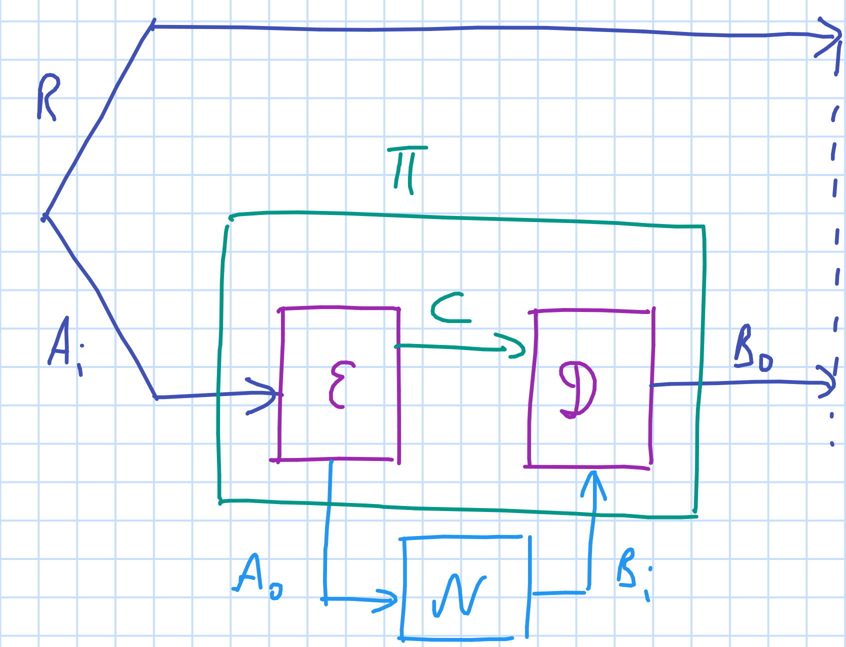


$\approx_\varepsilon \phi^+_{B_o R}$

Allow for some assistance helping encoder / decoder

(NB: 1-LOCC assistance is for free! [Barnum et al. '00])

Equivalently:



bipartite operation

$\Pi_{A_i B_i \to A_o B_o}$

encodes assistance C

Type of assistance controlled by properties of Π.

Quantum capacity: 1-LOCC assistance is free.

But LOCC is hard to characterize! [Chitambar et al. '14]

IDEA Relax the problem to make it easier...

Nice candidate: PPT-preserving operations

$$\rho_{A'A_i;B'B_i}^{T_{B'B_i}} \geq 0 \Rightarrow \left(\Pi_{A_iB_i \to A_oB_o}\, \rho_{A'A_i;B'B_i}\right)^{T_{B'B_o}} \geq 0$$

$\Leftrightarrow$ Choi state of $\Pi_{A_iB_i \to A_oB_o}$ is PPT wrt $A_iA_o | B_iB_o$

DEF PPT-assisted channel fidelity

Let $\mathcal{N}: A_o \to B_i$ be a channel, $k = |A_i| = |B_o|$ the code size:

$$F_{PPT}(\mathcal{N}, k) = \sup_{\Pi\ PPT} \langle \phi^+ |_{B_oR}\ \Pi_{A_iB_i \to A_oB_o} \circ \mathcal{N}\left(\phi^+_{A_iR}\right) |\phi^+\rangle_{B_oR}$$

DEF PPT-assisted one-shot ε-quantum capacity

$$Q^{(1)}_{PPT}(\mathcal{N}, \varepsilon) = \log \max \left\{ k \in \mathbb{N} : F_{PPT}(\mathcal{N}, k) \geq 1-\varepsilon \right\}$$

DEF PPT-assisted quantum capacity

$$Q_{PPT}(\mathcal{N}) = \lim_{\varepsilon \to 0} \lim_{n \to \infty} \frac{1}{n} Q^{(1)}_{PPT}\left(\mathcal{N}^{\otimes n}, \varepsilon\right)$$

1-LOCC $\subseteq$ PPT $\Rightarrow$ $\boxed{Q(\mathcal{N}) \leq Q_{PPT}(\mathcal{N})}$

LEM Semidefinite program for $F_{PPT}(\mathcal{N},k)$

[Leung, Matthews '15]

$$F_{PPT}(\mathcal{N},k) = \max \ \mathrm{tr}\, \tau^{\mathcal{N}}_{AB} W_{AB}$$

$$\text{s.t.} \quad 0 \le W_{AB} \le \rho_A \otimes \mathbb{1}_B$$

$$\mathrm{tr}\, \rho_A = 1$$

$$-\frac{1}{k} \rho_A \otimes \mathbb{1}_B \le W_{AB}^{T_B} \le \frac{1}{k} \rho_A \otimes \mathbb{1}_B$$

Straightforward to plug this into [Wang et al. '19]

$$Q^{(1)}_{PPT}(\mathcal{N},\varepsilon) = \sup\{k : F_{PPT}(\mathcal{N},k) \ge 1-\varepsilon\}$$

$$= -\log \min. \ m$$

$$\text{s.t.} \quad \mathrm{tr}\, \tau^{\mathcal{N}}_{AB} W_{AB} \ge 1-\varepsilon$$

$$0 \le W_{AB} \le \rho_A \otimes \mathbb{1}_B$$

$$\mathrm{tr}\, \rho_A = 1$$

$$-m\, \rho_A \otimes \mathbb{1}_B \le W_{AB}^{T_B} \le m\, \rho_A \otimes \mathbb{1}_B$$

PROBLEM This is <u>not</u> an SDP because of the bilinear $m\rho_A$ terms.

IDEA Relax the program for $Q^{(1)}_{PPT}$ to get an upper bound!

$$Q^{(1)}_{PPT}(\mathcal{N},\varepsilon) = -\log \min \Big\{ m : \operatorname{tr} \tau^{\mathcal{N}}_{AB} W_{AB} \geq 1-\varepsilon,\ \operatorname{tr} \rho_A = 1$$
$$0 \leq W_{AB} \leq \rho_A \otimes \mathbb{1}_B,$$
$$-\rho_A \otimes \mathbb{1}_B \leq \frac{1}{m} W_{AB} \leq \rho_A \otimes \mathbb{1}_B \Big\}$$

DEF $\Gamma(\mathcal{N}) = \max \operatorname{tr} \tau^{\mathcal{N}}_{AB} R_{AB}$

$$\text{s.t. } -\rho_A \otimes \mathbb{1}_B \leq R^{T_B}_{AB} \leq \rho_A \otimes \mathbb{1}_B$$
$$R_{AB}, \rho_A \geq 0, \quad \operatorname{tr} \rho_A = 1$$

THM $$Q^{(1)}_{PPT}(\mathcal{N},\varepsilon) \leq \log \Gamma(\mathcal{N}) - \log(1-\varepsilon)$$

[Wang et al. '19]

<u>Proof:</u> $$Q^{(1)}_{PPT}(\mathcal{N}) + \log(1-\varepsilon) = \log \frac{1-\varepsilon}{m} \leq \log \frac{\operatorname{tr} \tau^{\mathcal{N}}_{AB} W_{AB}}{m}$$
$$= \log \operatorname{tr} \tau^{\mathcal{N}}_{AB} R_{AB}$$
$$\leq \log \Gamma(\mathcal{N}) \qquad \square$$

SDP formulation of $\Gamma(\mathcal{N})$:

PRIMAL PROGRAM

$$\text{max. } \operatorname{tr} \tau_{AB}^{\mathcal{N}} R_{AB}$$

$$\text{s.t. } -\rho_A \otimes \mathbb{1}_B \leq R_{AB}^{T_B} \leq \rho_A \otimes \mathbb{1}_B$$

$$R_{AB}, \rho_A \geq 0, \ \operatorname{tr} \rho_A = 1$$

DUAL PROGRAM

$$\text{min. } \mu$$

$$\text{s.t. } (V_{AB} - Y_{AB})^{T_B} \geq \tau_{AB}^{\mathcal{N}}$$

$$V_A + Y_A \leq \mu \mathbb{1}_A$$

$$V_{AB}, Y_{AB} \geq 0$$

$\boxed{\text{PROP}}$ $\Gamma(\mathcal{N})$ is multiplicative: $\Gamma(\mathcal{N}_1 \otimes \mathcal{N}_2) = \Gamma(\mathcal{N}_1)\Gamma(\mathcal{N}_2)$

Proof uses SDP duality. [Wang et al. '19]

$\boxed{\text{THM}}$ $Q(\mathcal{N}) \leq \log \Gamma(\mathcal{N}) \leq \log \|\vartheta \circ \mathcal{N}\|_\diamond$

Proof: $Q(\mathcal{N}) \leq Q_{PPT}(\mathcal{N})$

$$= \lim_{\varepsilon \to 0} \lim_{n \to \infty} \frac{1}{n} Q_{PPT}^{(1)}(\mathcal{N}^{\otimes n}, \varepsilon)$$

$$\leq \lim_{\varepsilon \to 0} \lim_{n \to \infty} \frac{1}{n} \left[\log \Gamma(\mathcal{N}^{\otimes n}) - \log(1-\varepsilon) \right]$$

$$= \log \Gamma(\mathcal{N}).$$

For $\Gamma(\mathcal{N}) \leq \|\vartheta \circ \mathcal{N}\|_\diamond$ use SDP formulations. $\square$

Bounds based on (anti-) degradability

PPT-based upper bounds are:

- ·) powerful ✓
- ·) computable using SDPs ✓✓
- ·) <u>BUT:</u> PPT-bounds don't "know" about no-cloning. ✗

<u>Recall:</u> a channel $\mathcal{N}: A \to B$ with comp. channel $\mathcal{N}^c: A \to E$ is

degradable, if $\exists\, \mathcal{D}: B \to E$ s.t. $\mathcal{N}^c = \mathcal{D} \circ \mathcal{N}$,

antidegradable, if $\exists\, \mathcal{A}: E \to B$ s.t. $\mathcal{N} = \mathcal{A} \circ \mathcal{N}^c$.

·) $\mathcal{N}$ degradable : $Q(\mathcal{N}) = Q^{(1)}(\mathcal{N})$ (coherent information)

·) $\mathcal{N}$ antidegradable: $Q(\mathcal{N}) = 0$ (due to no-cloning)

<u>Zero-capacity channels:</u>

1) PPT-channels: $Q(\mathcal{N}) = 0 = Q_2(\mathcal{N})$

<u>But:</u> $P(\mathcal{N}) > 0$ (e.g. Horodecki channels)

2) Antidegradable channels: $Q(\mathcal{N}) = 0 = P(\mathcal{N})$

But: $Q_2(\mathcal{N}) > 0$ (e.g. erasure channels)

3)

$\boxed{\text{IDEA}}$ If channel $\mathcal{N}$ is ε-"close" to being degradable,

then $Q(\mathcal{N}) \le Q^{(1)}(\mathcal{N}) + f(\varepsilon)$. [Sutter et al. '17]

This is e.g. always satisfied for "low-noise" channels

satisfying $\|\mathcal{N} - \mathrm{id}\|_\diamond \le \varepsilon$ [FL, Leung, Smith '18]

Bottleneck inequality: $Q(\mathcal{N} \circ \mathcal{M}) \le \min\{Q(\mathcal{N}), Q(\mathcal{M})\}$

$\boxed{\text{IDEA}}$ Find "additive extension" $\mathcal{T}$ of $\mathcal{N}$ with $Q(\mathcal{T}) = Q^{(1)}(\mathcal{T})$

and $\mathcal{N} = \mathcal{R} \circ \mathcal{T}$ for some channel $\mathcal{R}$.

$\Rightarrow \quad Q(\mathcal{N}) \le Q^{(1)}(\mathcal{T})$. [Smith, Smolin '08]

In particular, write $\mathcal{N} = \sum_i p_i \mathcal{N}_i$ where each $\mathcal{N}_i \in \mathrm{DEG}$.

Define $\mathcal{T} = \sum_i p_i \mathcal{N}_i \otimes |i\rangle\langle i|$ "flagged deg. extension".

$\Rightarrow \quad Q(\mathcal{N}) \le \sum_i p_i Q^{(1)}(\mathcal{N}_i)$

- Generalizes no-cloning bounds $\left(Q(\mathcal{N}) = 0 \text{ for } \mathcal{N} \in \mathrm{ADG}\right)$
- Can be extended to include $\mathcal{N}_i \in \mathrm{ADG}$.

[FL, Datta, Smith '18]

References

- Barnum H., E. Knill, M. Nielsen, IEEE TIT 46 (4), pp. 1318–1329 (2000), arXiv: quant-ph/9809010
- Chitambar E. et al., Comm. Math. Phys. 328 (1), pp. 303–326 (2014), arXiv: 1210.4583
- Holevo A., R. Werner, PRA 63 (3), 032312 (2001), arXiv: quant-ph/9912067
- Horodecki M., P. Horodecki, R. Horodecki, PRL 80, pp. 5239–5242 (1998), arXiv: quant-ph/9801069
- Leditzky F., N. Datta, G. Smith, IEEE TIT 64 (7), pp. 4689–4708 (2018), arXiv: 1701.03081
- Leditzky F., D. Leung, G. Smith, PRL 120, 160503 (2018), arXiv: 1705.04335
- Leung D., W. Matthews, IEEE TIT 61 (8), pp. 4486–4499 (2015), arXiv: 1406.7142
- Smith G., J. Smolin, IEEE ITW, 2008, arXiv: 0712.2471
- Sutter D. et al., IEEE TIT 63 (12), pp. 7832–7844 (2017), arXiv: 1412.0980
- Wang X., K. Fang, R. Duan, IEEE TIT 65 (4), pp. 2583–2592 (2019) arXiv: 1709.00200
- Watrous J., Th. Comp. 5, pp. 217–238 (2009), arXiv: 0901:4709