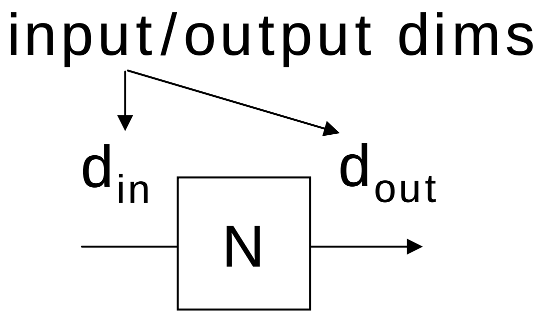
Continuity of channel capacities

0810.4931
L, Smith



A capacity of a channel (e.g. C(N)) is a function taking each N to a real number.

If two channels are close to one another under some distance measure, should their capacities be similar?

Continuity -- isn't it obvious ?

- Classical capacity of a classical channel

  Yes, expression is convex and single-letterized (with compact domain) in the input distribution

- Capacities of a quantum channel

  Many only have expression as an optimization over unbounded number of channel uses

  Even if $N \approx M$, $N^{\otimes n}$ & $M^{\otimes n}$ are very different

Consider classical messages first ...

Shannon's noisy coding theorem

$$C(N) = \max_{p(x)} I(X:Y) = \max_{p(x)} I(X:N(X))$$

HSW Theorem:

$$C(N) = \lim_{n\to\infty} \max_{p_x,\rho_x} 1/n \ S(X:B_1B_2 \cdots B_n)$$

$$\text{eval on } \sum_x p_x \underbrace{|x\rangle\langle x|}_{X} \otimes \underbrace{N^{\otimes n}(\rho_x)}_{B_1B_2 \cdots B_n}$$

Continuity of C(N) for classical channels:

$$C(N) = \max_X \boxed{H(X)+H(N(X))-H(XN(X))} = \max_X f(X,N)$$

For 2 channels N_1 and N_2,

the difference between $C(N_1)$ & $C(N_2)$ is caused by

- the difference between N_1 & N_2,
- also that between the optimal X_1 & X_2

We first remove this problem ...

Continuity of C(N) for classical channels:

$$C(N) = \max_X \boxed{H(X)+H(N(X))-H(XN(X))} = \max_X f(X,N)$$

For 2 channels N_1 and N_2,

Let X_i^{op} be optimal input distribution for N_i:

$\therefore C(N_1) \leq C(N_2) + \text{approx}$

$C(N_1) = f(X_1^{op},N_1)$ $\longleftrightarrow$ ($\approx$ equal) $\longleftrightarrow$ $f(X_1^{op},N_2)$

$\vee\!\mid$ $\qquad\qquad$ $\mid\!\wedge$

$f(X_2^{op},N_1)$ $\longleftrightarrow$ (if $\forall X$ f(X,N) cts in N these are $\approx$ equal) $\longleftrightarrow$ $f(X_2^{op},N_2) = C(N_2)$

$\therefore C(N_1) \geq C(N_2) - \text{approx}$

$$\therefore \; | C(N_1) - C(N_2) | \leq \max_X | f(X,N_1) - f(X,N_2) |$$

Continuity of C(N) for classical channels:

$C(N) = \max_X \boxed{H(X)+H(N(X))-H(XN(X))} = \max_X f(X,N)$

$| C(N_1) - C(N_2) | \leq \max_X | f(X,N_1) - f(X,N_2)|$

When does $N_1 \approx N_2$ imply $\forall X$ $|f(X,N_1) - f(X,N_2)|$ small?

(a) Want $N_1 \approx N_2$ implies $\forall X$ $XN_1(X) \approx XN_2(X)$

Take $||N_1-N_2|| = \max_X||XN_1(X)-XN_2(X)||_{tr}$

(b) Want f(X,N) is smooth in N:

$\Delta f \leq | H(N_1(X))-H(N_2(X)) | + | H(XN_1(X))-H(XN_2(X)) |$

$\leq ||N_1(X)-N_2(X)||_{tr} \log d_{out} + ||XN_1(X)-XN_2(X)||_{tr} \log d_{in}d_{out}$

$+ 2\,\eta(||N_1-N_2||)$ by Fannes inequality $(\,\eta(t)=-t \log t\,)$

$\leq 3\,||N_1-N_2|| \log d + 2\,\eta(||N_1-N_2||)$ where $d = \max(d_{in,out})$

Continuity of C(N) for quantum channels:

f(.,N)

cannot bound $f(.,N_1)-f(.,N_2)$

$$C(N) = \lim_{n\to\infty} \max_{p_x,\rho_x} \frac{1}{n} [S(X)+S(B_1\cdots B_n)-S(XB_1\cdots B_n)]$$

evaluated on $\sum_x p_x|x\rangle\langle x| \otimes N^{\otimes n}(\rho_x)$

Mimic continuity argument for classical channels:

(1) Use the diamond norm (cb trace norm):

$$||N_1-N_2||_\diamond := \max_\rho || I\otimes N_1(\rho) - I\otimes N_2(\rho) ||_{tr}$$

(2a) $\sum_x p_x|x\rangle\langle x| \otimes N_1^{\otimes n}(\rho_x)$ and $\sum_x p_x|x\rangle\langle x| \otimes N_2^{\otimes n}(\rho_x)$

can be "$n||N_1-N_2||_\diamond$" apart.

(2b) evaluating $S(B_1\cdots B_n)$ on the two states above

Fannes ineq bounds the difference as

"log dim" * distance of the two states + $\eta()$

$\log d_{out}^n = n \log d$ $\quad\quad n||N_1-N_2||$

Solution: tighter bound on entropy difference between two n-use output states.

<u>Main lemma [continuity of output entropy]:</u>

Let N,M: A $\rightarrow$ B be quantum channels, d=dim(B).

R reference system. If $||N-M||_\diamond \leq \varepsilon$, $\forall\ \rho_{RA^{\otimes n}}$,

$$| S(I \otimes N^{\otimes n}(\rho)) - S(I \otimes M^{\otimes n}(\rho)) | \leq n\ [4\ \varepsilon \log d + 2H(\varepsilon)]$$

same state

only 1 factor of n

The proof only requires:

- the telescopic sum,
- the triangular inequality, and

* the Fannes-Alicki inequality (quant-ph/0312081)

$$|S(K|L)_\rho - S(K|L)_\sigma| \leq \log[\dim(K)]\ |\rho-\sigma|_{tr} + \ldots$$

no L

Main lemma [continuity of output entropy]:

Let N,M: A $\rightarrow$ B be quantum channels, d=dim(B).

R reference system. If $||N-M||_\diamond \leq \varepsilon$, $\forall\ \rho_{RA^{\otimes n}}$,

$| S(I\otimes N^{\otimes n}(\rho)) - S(I\otimes M^{\otimes n}(\rho)) | \leq n\ [4\ \varepsilon \log d + 2H(\varepsilon)]$

Proof: Let $\sigma_k = I \otimes N^{\otimes k} \otimes M^{\otimes n-k} (\rho)$

If $|S(\sigma_k)-S(\sigma_{k-1})| \leq 4\ \varepsilon \log d + 2\ H(\varepsilon)$ ← now prove this

then $| S(I\otimes N^{\otimes n}(\rho)) - S(I\otimes M^{\otimes n}(\rho)) |$

$= |S(\sigma_n)-S(\sigma_0)|$

$= |\sum_{k=1}^n S(\sigma_k)-S(\sigma_{k-1})|$ telescopic sum

$= \sum_{k=1}^n |S(\sigma_k)-S(\sigma_{k-1})|$ triangular ineq

$\leq n\ [4\ \varepsilon \log d + 2H(\varepsilon)]$

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Proof: Let $\sigma_k = I \otimes N^{\otimes k} \otimes M^{\otimes n-k} (\rho)$

$|S(\sigma_k)-S(\sigma_{k-1})|$

$= |S(CB_1 \ldots B_n)_{\sigma\ k} - S(CB_1 \ldots B_n)_{\sigma\ k-1}|$ inserting 0

$= |S(CB_1 \ldots B_n)_{\sigma\ k} - S(CB_1 .. B_{k-1}B_{k+1} .. B_n)_{\sigma\ k}$

$\quad + S(CB_1 .. B_{k-1}B_{k+1} .. B_n)_{\sigma\ k} - S(CB_1 \ldots B_n)|_{\sigma\ k-1}$

$= |S(CB_1 \ldots B_n)_{\sigma\ k} - S(CB_1 .. B_{k-1}B_{k+1} .. B_n)_{\sigma\ k}$

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σ_k & σ_{k-1}
differ only in B_k

Main lemma [continuity of output entropy]:

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R reference system. If $||N-M||_\diamond \leq \varepsilon$, $\forall\ \rho_{RA^{\otimes n}}$,

$$| S(I\otimes N^{\otimes n}(\rho)) - S(I\otimes M^{\otimes n}(\rho)) | \leq n\ [4\ \varepsilon \log d + 2H(\varepsilon)]$$

Proof: Let $\sigma_k = I \otimes N^{\otimes k} \otimes M^{\otimes n-k} (\rho)$

$$|S(\sigma_k)-S(\sigma_{k-1})|$$

$$= |S(CB_1 \ldots B_n)_{\sigma k} - S(CB_1 \ldots B_n)_{\sigma\ k-1}|$$

$$= |S(CB_1 \ldots B_n)_{\sigma k} - S(CB_1 .. B_{k-1}B_{k+1} .. B_n)_{\sigma k}$$

$$+ S(CB_1 .. B_{k-1}B_{k+1} .. B_n)_{\sigma\ k-1} - S(CB_1 \ldots B_n)_{\sigma\ k-1}|$$

$$= |S(B_k|CB_1 .. B_{k-1}B_{k+1} .. B_n)_{\sigma k} - S(B_k|CB_1 .. B_{k-1}B_{k+1} .. B_n)_{\sigma\ k-1}|$$

$\leq 4\ ||\sigma_k - \sigma_{k-1}||_{tr} \log d + \ldots$ thanks to Alicki-Fannes!

$\leq 4\ ||N-M||_\diamond \log d + \ldots$ ← independent of dim of system being conditioned on!!

$\leq 4\ \varepsilon \log d + 2\ H(\varepsilon)$

Main lemma [continuity of output entropy]:

Let N,M: A $\rightarrow$ B be quantum channels, d=dim(B).

R reference system. If $||N-M||_\diamond \leq \varepsilon$, $\forall\ \rho_{RA^{\otimes n}}$,

$| S(I \otimes N^{\otimes n}(\rho)) - S(I \otimes M^{\otimes n}(\rho)) | \leq n\ [4\ \varepsilon \log d + 2H(\varepsilon)]$

Plug in the following:

f(.,N)

$$C(N) = \lim_{n \rightarrow \infty} \max_{p_X, \rho_X} \frac{1}{n} [S(X)+S(B_1 \cdots B_n)-S(XB_1 \cdots B_n)]$$

evaluated on $\sum_x p_x |x\rangle\langle x| \otimes N^{\otimes n}(\rho_x)$

$| C(N_1) - C(N_2) | \leq \max_X | f(X,N_1) - f(X,N_2)|$

Get corollary 1: If $|| N_1 - N_2 ||_\diamond \leq \varepsilon$, then

$|C(N_1)-C(N_2)| \leq 8\ \varepsilon \log d + 4\ H(\varepsilon)$.

Buy 1 get 2 free:

- Quantum capacity (Lloyd-Shor-Devetak)

$$Q(N) = \lim_{n\to\infty} \max_{\psi} 1/n \; I^{coh}(R>B_1B_2 \cdots B_n)$$

evaluated on $I\otimes N^{\otimes n}(\psi_{RA_1A_2 \cdots A_n})$

Corollary 2: If $|| N_1 - N_2 ||_\diamond \leq \varepsilon$, then
$|Q(N_1)-Q(N_2)| \leq 8\, \varepsilon \log d + 4\, H(\varepsilon)$.

- Private classical capacity (Smith-Smolin-Winter)

$$C_p(N) = \lim_{n\to\infty} \max_{p_x,\rho_x} 1/n \; [I(X:B_1B_2 \cdots B_n)-I(X:E_1E_2 \cdots E_n)]$$

eval on $\sum_x p_x|x\rangle\langle x| \otimes U^{\otimes n}(\rho_{x\; RA_1A_2 \cdots A_n})$

Corollary 3: If $|| N_1 - N_2 ||_\diamond \leq \varepsilon$, then
$|C_p(N_1)-C_p(N_2)| \leq 16\, \varepsilon \log d + 8\, H(\varepsilon)$.

Now what about Q_2 [quantum capacity assisted by free 2-way classical communication]?

There's no capacity expression, though $Q_2(N) = E(N)$ (entanglement capacity of the channel)

Guifre Vidal proved distillable entanglement is continuous. That doesn't imply anything for the entanglement capacity of a channel itself.

The result is not applicable, but the idea is.

If two distillable states ρ_1, ρ_2 are similar, n copies of ρ_1 can be converted into $\approx$ n copies ρ_2 with LOCC. So, distillable entanglement of ρ_1 cannot be much less than that of ρ_2 . Same with ρ_1 and ρ_2 interchanged.

Such conversion works for channels too!

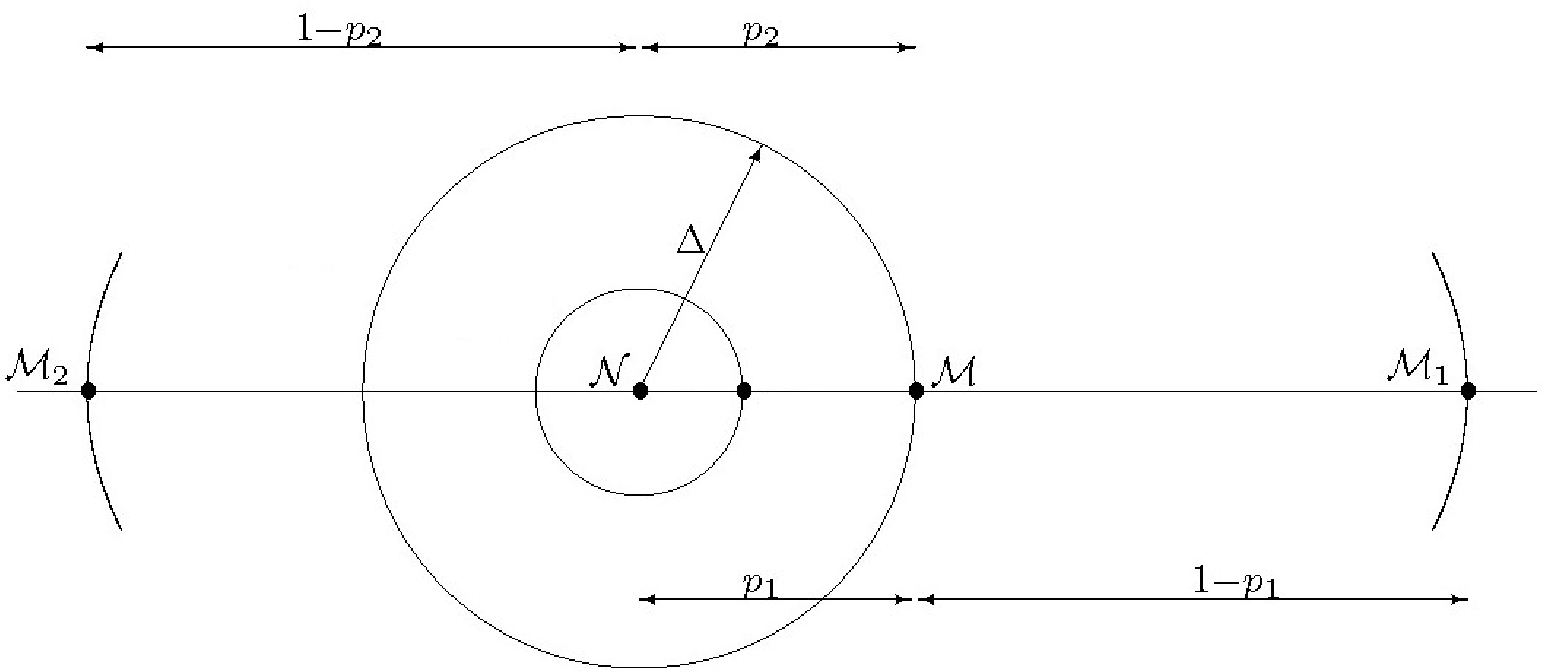
Continuity of Q_2 in the interior of $\{Q_2(N)>0\}$

Given channels M, N with $Q_2>0$, $\exists$ M_1, M_2 such that:

$M = p_1 M_1 + (1-p_1) N$

$N = p_2 M_2 + (1-p_2) M$

$d = \min(d_{in}, d_{out})$



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$d=\min(d_{in},d_{out})$

I. Simulate M using N:

(1) Simulate M using M_1, N, & free CC

Receiver tosses n coins with bias p_1, tells sender with free CC, the ith coin toss decides whether M_1 or N is used for the ith simulation of M

$$np_1 M_1 + n(1-p_1) N \geq n M$$

(2) Simulate M_1 using I using N

$$np_1 (\log d/Q_2(N)) N \geq np_1 I \geq np_1 M_1$$

Compose (1) & (2): $n [p_1 \log d/Q_2(N)+(1-p_1)] N \geq n M$

$$\therefore [p_1 \log d/Q_2(N)+(1-p_1)] Q_2(N) \geq Q_2(M)$$

Rearranging: $p_1 (\log d-Q_2(N)) \geq Q_2(M)-Q_2(N)$

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I. Simulate M using N:

$$p_1 (\log d-Q_2(N)) \geq Q_2(M)-Q_2(N)$$

II. Simulate N using M:

Applying the same argument to the blue equation:

$$p_2 (\log d-Q_2(M)) \geq Q_2(N)-Q_2(M)$$

Thus, $|Q_2(N)-Q_2(M)| \leq \max(p_1,p_2) * \log d$

Continuity of Q_2 in the interior of $\{Q_2(N)>0\}$

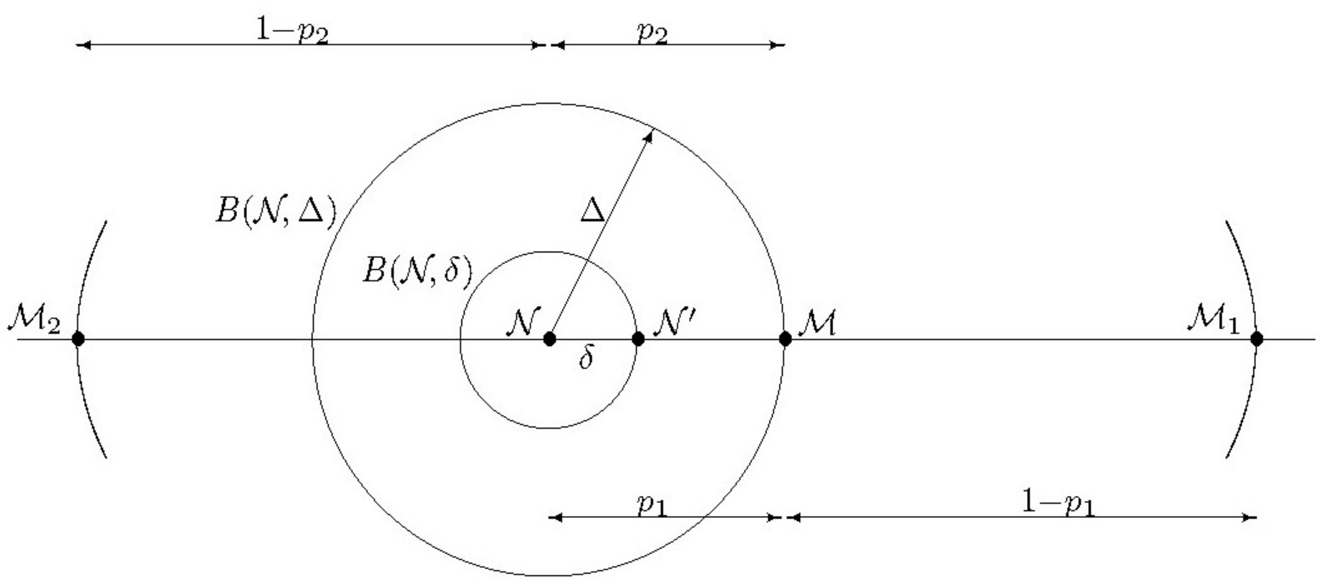
Given channels M, N with $Q_2>0$, $\exists$ M_1, M_2 such that:

$M = p_1 M_1 + (1-p_1) N$

$N = p_2 M_2 + (1-p_2) M$

$d=\min(d_{in},d_{out})$

$|Q_2(N)-Q_2(M)| \leq \max(p_1,p_2) * \log d$



Replace M by N', then p_1, $p_2 \to 0$ & $|Q_2(N') - Q_2(N)| \to 0$

Same argument holds for $Q_B(N)$ (assisted by free back classical communication).

Q_B of the erasure channel is continuous in the erasure probability p for all p.

So, is continuity "obvious" ?

1. There are pairs of channels (N_1^n, N_2^n) s.t. as $n \to \infty$,

$$|| N_1^n - N_2^n ||_\diamond \to 0, \text{ but } |C(N_1^n) - C(N_2^n)| = 1$$

All the channels take a space spanned by $\{|1\rangle, |2\rangle, \ldots\}$ to $\{|0\rangle, |1\rangle, |2\rangle, \ldots\}$

$\forall$ n, $N_1^n = N$, $N(\rho) = tr(\rho)\ |0\rangle\langle 0|$. $C(N) = 0$.

$N_2^n = \left(1 - \frac{1}{\log n}\right)\mathcal{N} + \frac{1}{\log n}\,\mathrm{id}_n$. $C(N_2^n) \geq 1$.

identity on $|1\rangle, \cdots, |n\rangle$, acts like N elsewhere

$$|| N_1^n - N_2^n ||_\diamond = || N - id_n ||_\diamond / \log n \leq 2 / \log n$$

A slightly different type of channels exhibit the same phenomena for Q(N)

So, is continuity "obvious" ?

2. For classical arbitrary varying channels with const input/output dimensions, the capacity (allowing LOCAL randomness) is not continuous when the capacity drops to zero.

3. Unresolved cases:

is $Q_{2 \text{ or } B}(N)$ continuous where $Q_{2 \text{ or } B}(N) = 0$?

is $D_{1 \text{ or } 2}(\rho)$ continuous where $D_{1 \text{ or } 2}(\rho) = 0$?