Some remarks on the additivity problem and the Hastings counterexample - part 2 -

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How to show the violation

We show that $\exists\Phi$ a quantum channel such that

$$S_{\min}(\Phi \otimes \bar{\Phi}) < S_{\min}(\Phi) + S_{\min}(\bar{\Phi})$$

Here, for $\rho \in \mathcal{M}_n$

$$\Phi(\rho) = \sum_{k=1}^{d} A_k \rho A_k^* \qquad \overline{\Phi}(\rho) = \sum_{k=1}^{d} \bar{A}_k \rho A_k^T$$

We show the following results.

1) For any channel Φ

$$S_{\min}(\Phi \otimes \bar{\Phi}) \lessapprox 2\log d - \frac{\log d}{d}$$

2) There exists a channel Φ such that

$$S_{\min}(\Phi) > \log d - \frac{h}{2d}.$$

Here, the constant h is independent of d or n.

Upper bound for $S_{\min}(\Phi \otimes \bar{\Phi})$

Hayden and Winter showed that for maximally entangled state $|\psi_M\rangle$

$$\langle\psi_M|(\Phi \otimes \overline{\Phi})(|\psi_M\rangle\langle\psi_M|)|\psi_M\rangle \geq \frac{1}{d}$$

Consider the worst case, where the eigenvalues are

$$\left\{ \frac{1}{d}, \underbrace{\frac{1}{d^2-1}\left(1-\frac{1}{d}\right), \cdots, \frac{1}{d^2-1}\left(1-\frac{1}{d}\right)}_{d^2-1} \right\}$$

This gives an upper bound for the entropy:

$$\begin{aligned} (\Phi \otimes \overline{\Phi})\left(|\psi_M\rangle\langle\psi_M|\right) &\leq -\frac{1}{d}\log\left(\frac{1}{d}\right) - \left(1-\frac{1}{d}\right)\log\left(\frac{1}{d(d+1)}\right) \\ &\approx 2\log d - \frac{\log d}{d} \end{aligned}$$

More on the largest eigenvalue

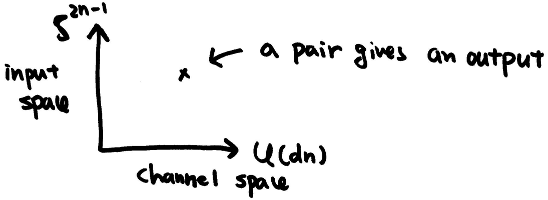
Hayden and Winter showed that

$$\langle\psi_M| \left(\Phi \otimes \overline{\Phi}\right) (|\psi_M\rangle\langle\psi_M|)|\psi_M\rangle \geq \frac{\dim(\text{input})}{\dim(\text{output}) \times \dim(\text{environment})}$$

If dim(input) is large, we get a good upper bound for $S_{\mathbf{min}}(\Phi \otimes \overline{\Phi})$. However, since the input space embedded in (output space) $\otimes$ (environment space) is relatively large (Stinespring dilation) it is difficult to get a nice lower bound for $S_{\mathbf{min}}(\Phi)$. I.e., the randomly chosen subspace is likely to pick up a state with small entanglement.

Event space

We set the product of the input space (only pure states) and the quantum channel space. We call it Ω.



1)The measure on the input space can be fixed to be the uniform measure on $\mathbb{S}^{2n-1}$, which we call σ_n.
2)The measure for the quantum channel space can be induced from the Haar measure on $\mathcal{U}(dn)$, which we call μ_{dn}. ($\to$ the next slide).

Random quantum channels

A quantum channels is written in the Kraus form.

$$\Phi(\rho) = \sum_{i=k}^{d} A_k \rho A_k^* \qquad \left(\sum_k A_k^* A_k = I \right)$$

Note that let W be a partial isometry defined as

$$W = \begin{pmatrix} A_1 \\ \vdots \\ A_d \end{pmatrix} : \mathbb{C}^n \to \mathbb{C}^n \otimes \mathbb{C}^d$$

We identify the space of channel as $\mathcal{U}(dn)$ with the Haar measure. (We denote the measure μ_{dn}.) Importantly, by Steinespring formula,

$$\Phi(|\psi\rangle\langle\psi|) = \mathrm{tr}_{\mathbb{C}^d}[W|\psi\rangle\langle\psi|W^*]$$

Note that $W|\psi\rangle$ is a uniform random vector on $\mathbb{S}^{2dn-1}$ for any fixed $|\psi\rangle$.

Two measures on Δ_d

Define two maps:

$$\begin{array}{rcl} H : \mathbb{S}^{2n-1} \times \mathcal{U}(dn) & \to & \Delta_d \\ (|\psi\rangle, \Phi) & \mapsto & \text{eigenvalues of } \Phi^C(|\psi\rangle\langle\psi|) \\ G : \qquad \mathbb{S}^{2dn-1} & \to & \Delta_d \\ |\tilde{\phi}\rangle & \mapsto & \text{eigenvalues of } \mathrm{Tr}_{\mathbb{C}^n}\left[|\tilde{\phi}\rangle\langle\tilde{\phi}|\right] \end{array}$$

Here, Δ_d is the simplex of d-dimensional probability distributions. Then, the two measure on Δ_d defined as push-forward by the above H and G are identical.

$$H^*(\sigma_n \times \mu_{dn}) = G^*(\sigma_{dn})$$

Idea of proof

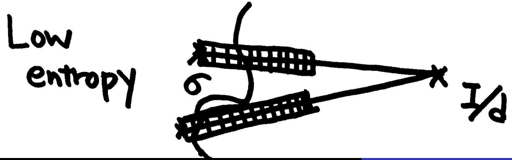
A contradiction argument:
"Suppose for any quantum channel there exists an input which gives a low entropy output."
We calculate the probability that outputs are close to the following states (We call this event $E \subset \Omega$):

$$\lambda\sigma + (1-\lambda)\frac{I}{d}$$

Here,

$$S(\sigma) < \log d - \frac{h}{d} \qquad \text{and} \qquad \frac{1}{2} \leq \lambda \leq 1$$

Importantly, E can be "defined" only by eigenvalues of outputs.



Wishart distribution

This is why, there exists $A \subset \Delta_d$ such that $E = H^{-1}(A)$. Also, remember that

$$H^*(\sigma_n \times \mu_{dn}) = G^*(\sigma_{dn})$$

So, by using the Wishart distribution for $\mathbb{S}^{2dn-1}$ we have

$$\begin{aligned} & \Pr(E) \\ = \; & \frac{1}{Z(n,d)} \int_A \prod_{1 \le i < j \le d} (p_i - p_j)^2 \prod_{i=1}^d p_i^{n-d} \, \delta\left(\sum_{i=1}^d p_i - 1 \right) dp \\ \lessapprox \; & \exp\left\{ (n-d) \sup\left\{ \sum_{i=1}^d \log(d\, p_i) : \{p_i\} \in A \right\} \right\} \end{aligned}$$

Lagrange method for upper bound of $\Pr(E)$

We (sort of) solve the following problem:

$$\max \sum_{i=1}^{d} \log(d\, p_i)$$

with constraints:

$$p_i = \lambda q_i + (1-\lambda)\frac{1}{d} \qquad i = 1, \cdots, d$$

Here,

$$S(\{q_1, \cdots, q_d\}) < \log d - \frac{h}{d} \qquad \text{and} \qquad \frac{1}{2} \le \lambda \le 1$$

This gives something like

$$\Pr(E) \lessapprox \exp\left\{ -n \log\left(\frac{h}{40}\right)\right\}$$

The speed of convergence as $n \to \infty$ depends on h.

Decomposition of unit vector

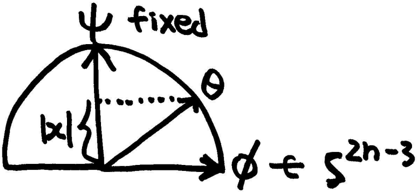
Fix a unit vector $|\psi\rangle \in \mathcal{S}^{2n-1}$. Then, the uniformly distributed unit vector $|\theta\rangle \in \mathcal{S}^{2n-1} \subset \mathbb{C}^n$ can be written as

$$|\theta\rangle = x|\psi\rangle + \sqrt{1-|x|^2}\,|\phi\rangle$$

Here, first $\langle\psi|\theta\rangle = x$ with

$$\Pr\{|\theta\rangle : |x|^2 = |\langle\psi|\theta\rangle|^2 > \gamma\} = (1-\gamma)^{n-1}.$$

and secondly $|\theta\rangle \in \mathcal{S}^{2n-3} \perp |\psi\rangle$ is uniformly distributed.
Importantly, x and $|\phi\rangle$ are independent random variables.



|x| is likely to be small !

But not too small.

Behavior of outputs

Assuming $d \ll n$

$$\Phi^C(|\theta\rangle\langle\theta|) = |x|^2 \underbrace{\Phi^C(|\psi\rangle\langle\psi|)}_{\text{low entropy output}} + (1-|x|^2) \underbrace{\Phi^C(|\phi\rangle\langle\phi|)}_{\text{close to } I/d \ (\star)}$$
$$+ \underbrace{\sqrt{1-|x|^2}\left(x\Phi^C(|\psi\rangle\langle\phi|) + \bar{x}\Phi^C(|\phi\rangle\langle\psi|)\right)}_{\text{small } (\star\star)}$$

i) $\star$ happens with high probability on Φ and ϕ
ii) $\star\star$ happens with high probability on ϕ
iii) $|x|^2 > \frac{1}{2}$ with probability $(\frac{1}{2})^{n-1}$
Therefore, we get a lower bounded;

$$\Pr(E) \geq \frac{1}{4} \times \left(\frac{1}{2}\right)^{n-1} = \frac{1}{4}\exp\{-(n-1)\log 2\}$$

Note that the speed of convergence as $n \to \infty$ is fixed.

Contradiction (with a rough calculation)

Assuming $d \ll n$,

$$\Pr(E) \lessapprox \exp\left\{-n \log\left(\frac{h}{40}\right)\right\}$$
$$\Pr(E) \gtrapprox \exp\{-n \log 2\}$$

A contradiction for $h > 80$.
Solving

$$2h = \log d$$

we get

$$d = \exp\{160\} \qquad \text{a big number!}$$

General case [Fukuda, King]

Suppose we have quantum channels induced by isometric embedding:

$$W : \mathbb{C}^n \to \mathbb{C}^d \otimes \mathbb{C}^m$$

Take increasing sequences $\{n_k, m_k,\}$ such that

$$\text{i)} \qquad r_1 \leq \frac{n_k}{m_k} \leq r_2 \qquad \text{for some constants } r_1, r_2 > 0$$

$$\text{or} \quad \text{ii)} \qquad \frac{n_k}{m_k} \to 0, \qquad \frac{m_k \log n_k}{n_k^{3/2}} \to 0$$

Then, there exists $h > 0$ and for large k there exists an isometric embedding W such that

$$\min_{\rho \in \mathcal{M}_n} S\left(\mathrm{Tr}_{\mathbb{C}^m}[W \rho W^*]\right) \geq \log d - h\left(\frac{n_k}{m_k d}\right)$$