Lec 9, 2010 (part 1)

Last time, we stated ① ② ③ are equivalent:

① \( \forall N_1, N_2 \) \( X''(N_1 \otimes N_2) = X''(N_1) + X''(N_2) \)
② \( \forall p_1, p_2 \) \( EF(p_1 \otimes p_2) = EF(p_1) + EF(p_2) \)
③ \( \forall N_1, N_2 \) \( S_{\min}(N_1 \otimes N_2) = S_{\min}(N_1) + S_{\min}(N_2) \)

Define \( X_p(N) = \max_{\text{given } \ell \in \{p_1, p_2\}} S(N|p_\ell) - \sum_{\ell \neq p} S(N|p_\ell) \)

Let ⑥ \( \forall p_1, p_2 N_1, N_2 \) \( X_{p_1 \otimes p_2}(N_1 \otimes N_2) = X_{p_1}(N_1) + X_{p_2}(N_2) \)

Plan: ① \( \Rightarrow \) ⑥ \( \Rightarrow \) ① \( \Rightarrow \) ①, ② \( \Leftrightarrow \) ②

simple, real, pain, missing, simple
1 ⇒ (3) Add of $X^{(i)}$ ⇒ Add of $\text{Smin}$

Given any $N_1$ & $N_2$, want $\text{Smin}(N_1 \otimes N_2) = \text{Smin}(N_1) \otimes \text{Smin}(N_2)$

Idea: construct $N_1'$ out of $N_1$, $N_2'$ out of $N_2$
and use $X^{(i)'}(N_1' \otimes N_2') = X^{(i)'}(N_1') \otimes X^{(i)'}(N_2')$

\[d'_{\text{out}} = \sum_{x=1}^{d'_{\text{out}}} U_x \eta U_x^+ = \frac{I}{d'_{\text{out}}}\]

\[N_1'(1 \times 1 \otimes \rho) = U_x N_1(\rho) U_x^+\]
Recall optimal ensemble has only pure states. Because of special structure of \( N_i \) the \( i \)th signal state should look like \( \left| x_i \right\rangle \left| y_i \right\rangle \). Also, to max \( X'' (N_i) \):

1. \( \max S(\text{average output}) \)
2. \( \min \text{ave} S(\text{individual outputs}) \)

This can be maximized by choosing \( x_i \) uniformly while taking \( \left| y_i \right\rangle \) the same. \( Y_i \)

\[ \text{entropy} = S(N_i(1_{Y_i} \times 1_{Y_i})) \]

\[ \text{where } f = 1_{Y_i} \times 1_{Y_i} \min S(N_i(f)) \]

Simultaneously optimized.
\[ \chi' (N_1) = \log(d_{\text{out}}') - \min_{14} S(N_1 (14,7<4.1)) \]

Similarly, construct \( N_2' \) out of \( N_2 \) in the same way.

(Note \( N_1, N_2 \) can have different input/output dimensions.)
So, consider $N_1 \otimes N_2$ (which is also $(N_1 \otimes N_2)'$): out. The optimal ensemble has $d_{out}^{12} \times d_{out}^{22}$ states of the form $|x_1 \rangle |x_2 \rangle |14 \rangle$, drawn uniformly, where $x_1 = 1, \ldots, d_{out}^{12}$, $x_2 = 1, \ldots, d_{out}^{22}$.

\[ f = 14 \times 41 \quad \min \quad S(N_1 \otimes N_2 (14)) \]

$Lives in the joint input spaces of$N_1$&$N_2$

\[ X^{(1)}(N_1 \otimes N_2) = \log d_{out}^{11} + \log d_{out}^{22} - \min \quad S(N_1 \otimes N_2 (14 \times 41)) \]
But \( \chi^{(1)}(N_1) = \log d_{\text{out}} - \min_{14_1} S(N_1(14_1 \times 4_1)) \)

\( \chi^{(1)}(N_2) = \log d_{\text{out}}^2 - \min_{14_2} S(N_2(14_2 \times 4_2)) \)

If (i) is true i.e \( \chi^{(1)} \) is additive for ANY 2 elements then it holds for \( N_1' \) & \( N_2' \)

\[ \chi^{(1)}(N_1' \cup N_2') = \chi^{(1)}(N_1') + \chi^{(1)}(N_2') \]

Cancelling the \( \log(d_{\text{out}}) \) terms & negating gives:

\[ \min_{14} S(N_1 \cup N_2(14_1 \times 4_1)) = \min_{14_1} S(N_1(14_1 \times 4_1)) + \min_{14_2} S(N_2(14_2 \times 4_2)) \]

And the above holds for ANY \( N_1 \) & \( N_2 \). i.e. (ii) is true.
Trying \( (3) \Rightarrow (0) \) or \( (3) \Rightarrow (1) \):

No luck...
Showing $\odot \iff \odot$:

Ingredients:

(i) Stinespring dilation / isometric extension of $N$:

Given $N$ (taking sys $A'$ to $B$)

$\exists \ U \ (\text{taking sys } A' \text{ to } BE)$

s.t. $N(\rho) = \text{tr}_E \ U \rho U^+$

Diagram:

$A' \ N \ B = U \ A' \ U^+$

$U$ isometry

$\text{tr}_E \ U \rho U^+$

Unitary degree of freedom on $U$

$U$ unitary
(ii) Matsumoto, Shimono, Winter:

Given $\rho$ and $N$, let $\{p_i, \psi_i\}$ be chosen pure

\[ \chi_{\rho}(N) = \max_{\{p_i, \psi_i\}} S(N(p)) - \sum_{i} p_i \, S(N(p_i)) \]

\[ = S(N(1)) - \min_{\{p_i, \psi_i\}} \sum_{i} p_i \, E(U_1 U_i) \]

\[ \leq \text{entanglement across } B \in \mathcal{F} \text{ of } \sum_{i} p_i \, U_1 U_i \times U_i U_1 U^T \]

\[ = U^T U = g[U]_E^B \]
Now, given $N_1, N_2, \rho_1, \rho_2$

Let $N_1 = U_1$

$N_2 = U_2$

arbitrary on the joint input spaces

\[
\chi_{\rho_1 \otimes \rho_2}(N_1 \otimes N_2) = \max \left\{ \begin{array}{l}
S(N_1 \otimes N_2(\rho_1 \otimes \rho_2)) \\
- \sum_{\xi_1 = \rho_1 \otimes \rho_2} \xi_1 S(N_1 \otimes N_2(\xi_1))
\end{array} \right\}
\]

// MSW

\[
S(N_1 \otimes N_2(\rho_1 \otimes \rho_2)) - EF(U_1 \otimes U_2(\rho_1 \otimes \rho_2))
\]

//

\[
S(N_1(\rho_1)) + S(N_2(\rho_2)) - EF(U_1(\rho_1)U_1^+ \otimes U_2(\rho_2)U_2^+)
\]
\[ x_{\rho_1}(N_1) + x_{\rho_2}(N_2) + \left[ \text{Ef}(U_1 f, u^+) + \text{Ef}(U_2 f_2 u^+) \right] \]

\[ - \text{Ef}(U_1 f, u_1^+ \otimes U_2 f_2 u_2^+) \]

1. \( \rho \) holds (Ef additivity) then \( \rho \) holds \( \forall N_1 N_2 f_1 f_2 \).

Converse: given \( \delta_1 \& \delta_2 \), we can always find \( f_1 f_2, U_1, U_2 \) s.t. \( \delta_1 = U_1 f_1 u_1^+ \), \( \delta_2 = U_2 f_2 u_2^+ \).

2. \( \rho \) \( \Rightarrow \) \( \rho \) But unfortunately, does not give \( \rho \) \( \Rightarrow \) \( \rho \) since in \( x^{(1)} \) we cannot control what \( f \) is.

Claim of \( \rho \) \( \Rightarrow \) \( \rho \) were made based on finding a channel for which one can control the optimal average output but no luck in my reading (with tight time constraints).