Moving on to transmit quantum data via quantum channels

As in the case to send classical data via a classical/quantum channel, channel noise degrades the transmitted state, and error correcting codes are used to suppress the degradation (at lower data rate).

Def: given quantum channel $N$, we say $R$ is an achievable rate if $\forall N, \exists (M = 2^{nR}, n)$ code transmitting an $M$-dim quantum state with $n$ uses of $N$, error $\leq \epsilon_n$ & $\epsilon_n \to 0$ as $n \to \infty$

Diagram:
- $\rho \rightarrow E^{(n)} \rightarrow N \rightarrow S^{(n)} \rightarrow \tilde{\rho} \approx \rho$
- $M = 2^{nR}$ dim
- $\epsilon_n \to 0$ as $n \to \infty$

Quantum Capacity $Q(N)$ = supremum overall achievable rates
Problem: how to define $\tilde{\rho} \equiv \rho$?

There are many different definitions, but luckily they agree to the same capacity. See eg 0311037 by Kretschmann & Werner.

We can demand any of the following:

1. $\left| \psi^{(14)} \right\rangle \rightarrow \begin{array}{c} 3(n) \end{array} \begin{array}{c} N_{\omega}^{(14)} \end{array} \begin{array}{c} 2(n) \end{array} \rightarrow \tilde{\rho} \begin{array}{c} \langle 41 \tilde{\rho} 14 \rangle \geq 1 - \varepsilon n \\
\end{array}$

2. $\forall \left| \psi^{(14)} \right\rangle$, $\langle 41 \tilde{\rho} 14 \rangle \geq 1 - \varepsilon n$

Can also replace $\langle 41 \tilde{\rho} 14 \rangle = 1 - \varepsilon n$ by $\| 14 \times 41 - \tilde{\rho} \| tr \leq \varepsilon n$
Def 1 revolves about "preserving" the input state traditionally the easier one to show / think about.

Def 2 revolves about "simulating" the identity channel on $2^R$ -dim Hilbert space.

When used w/ the trace distance criteria, Def 2 tells us how good the simulation is, measured by the trace distance of the "Choi–Jamiołkowski" states of the simulating & the simulated channels.

\[
\text{[CJ state of } N \text{ is just } |x\rangle \langle x| \text{ for } |x\rangle = \max_{|\psi\rangle} \text{ state]}
\]

This completely specifies the channel $N$. 
2b) tells us the distance between the simulated & the simulating channel in the "diamond norm":

\[ \| N_1 - N_2 \|_D = \max_{\tilde{\Omega}} \| \tilde{\Omega} N_1(14x14) - \tilde{\Omega} N_2(14x14) \|_{tr} \]

2 channels having small diamond norm is the most demanding way to say they're similar.

Aside: some researchers use the "C6-norm"

Besides the Stinespring dilation & the CJ state, a channel $N$ can also be written in the "bra-ket form"

\[ N|\psi\rangle = \sum_k A_k \rho A_k^\dagger \quad \text{ where } \sum_k A_k^\dagger A_k = I \]
$$N^*(\eta) = \sum_{k} A_k^\dagger \eta A_k$$ is called the "adjoint" of $N$.

If $N$ evolves state $\psi$ in the Schrödinger picture then $N^*$ evolves observables $\hat{\eta}$ in the Heisenberg picture.

$$\|N_1 - N_2\|_\alpha = \|N_1^* - N_2^*\|_{cb}$$

Luckily, if $N_1 \approx I$, $N_1^* \approx I$

so the diamond & cb norm are similar when measuring the accuracy of approx the identity.
On: What is an \((M, n)\) code for quantum data?

**In the classical case:**

- Error "sphere" for different codewords do not overlap.

**Better view:**

- Codewords form a subspace \(C\) & "likely errors" translate \(C\) to orthogonal spaces.

**Quantum case**

- Exactly like this
- Except you can have different errors \(E_1\) & \(E_2\) taking \(C\) to the same space.
- When this happens \(C\) is called "degenerate."
To be concrete:

9 bit degenerate code:

10\rangle_L \sim (10000 + 11111) \otimes 3

11\rangle_L \sim (10000 - 11111) \otimes 3

Assuming error on 2 or more qubits "unlikely", then the likely errors are linear combinations of Pauli operators on up to 1 qubit.

\bar{z}_1, \bar{z}_2, \bar{z}_3 all act the same on C (same for \bar{z}_4, \bar{z}_5, \bar{z}_6, \bar{z}_7, \bar{z}_8, \bar{z}_9).

\bar{z}_1, \bar{z}_4, \bar{z}_7, X_1-q Take C to ortho spaces, i.e. They're "distinguishable" thus correctable.

This code C is "+1" eigen space of commuting operators:

X_1 X_2 X_3 X_4 X_5 X_6, X_4 X_5 X_6 X_7 X_8 X_9, \bar{z}_1 \bar{z}_2, \bar{z}_1 \bar{z}_3, \bar{z}_4 \bar{z}_5, \bar{z}_5 \bar{z}_6, \bar{z}_7 \bar{z}_8, \bar{z}_8 \bar{z}_9
5 bit nondegenerate code:

+1 eigenspace of \( X_1 Z_2 Z_3 X_4 \), \( X_2 Z_3 Z_4 X_5 \), \( X_3 Z_4 Z_5 X_1 \), \( X_4 Z_5 Z_1 X_2 \)

Back to quantum capacity. Reference system (not achievable R)

Let\[ A \begin{array}{c} N \end{array} B \] and\[ A \begin{array}{c} N \end{array} B \]

called the coherent inf

\[ \text{Def: } Q^{(1)}(N) = \max_{14} \left( I_{C}(R > B) \right)_{\text{ION}(14 \times 41)} \]

Where \( I_{C}(R > B) = [S(B) - S(RB)] \) evaluated on \( I \otimes N(14 \times 41) \)

\[ \text{Def: } Q^{(1)}(N) = \frac{1}{n} Q^{(1)}(N^{\otimes n}) \]
The LSD thm:

\[ Q(N) = \sup_n Q^{(n)}(N) \]

In other words:

\[ \begin{array}{ccc}
A_1 & & B_1 \\
A_2 & & B_2 \\
\vdots & & \vdots \\
A_n & & B_n \\
\end{array} \]

\[ \mathbb{I}_R \otimes N^{\otimes n} (\chi_{\chi_{\chi_{\chi_{\chi}}}}) \]

- Calculate \( S(B_1 \cdots B_n) - S(R \cdot B_1 \cdots B_n) \)
- Divide by \( n \)
- Sub over \( n \).
So, what is $I(c \langle R | B_1 \cdots B_n \rangle)$?

\[ I(c \langle R | B \rangle) = S(B) - S(B_R) = S(E_R) - S(E) \]
\[ = \frac{1}{2} \left( [S(B) - S(B_R) + S(R)] + [S(E_R) - S(E) - S(R)] \right) \]
\[ = \frac{1}{2} \left[ S(B;R) - S(E;R) \right] \]
When transmitting classical data, we max $S(B:R)$ (evaluated on $\sum_x p_x R x \times x \times 1 \times N(p_x)$)

For quantum data, not only we need to max Bob correlation with $R$, he needs to take out a portion of his data that's correlated with Eve. The QMI counts all correlation (classical + quantum) and the classical portion is quantified by $SR=E$. 
\text{eg. let } N = 1 \text{ qbit + 1 \text{ qbit}. Stinespring's dilation}

\begin{align*}
\text{Input:} & \quad \begin{pmatrix} 100 \oplus 111 \end{pmatrix}_{AB} \\
\text{Output:} & \quad \begin{pmatrix} 100 \oplus 111 \end{pmatrix}_{R} \otimes \begin{pmatrix} 1000 \oplus 1111 \end{pmatrix}_{R'} \\
& \quad \quad \text{s}(R \oplus B) = 2 \\
& \quad \quad \text{s}(R' \oplus B') = 3 \\
& \quad \quad \text{s}(R' \oplus E) = 1 \\
& \quad \quad \text{s}(RR' \oplus BB') = 3, \quad \text{s}(RR' \oplus E) = 1 \\
I_c(\text{RR'} \rightarrow \text{BB'}) = \frac{1}{2} (3 - 1) = 1 \text{ (expected).}
\end{align*}
Properties of the coherent information $I(R;B)$:

1. Invariant under local unitary on $R$ & $B$
2. Invariant under attaching ancillas in pure states
3. Non-increasing under local operation on $B$’s side

Proof by monotonicity of QMI:

\[ I(R;B) \geq I(R;B') \]

Since evaluated on $\rho = \text{tr}_B[I\otimes \rho(R)]$

\[ I(R;B) \geq I(R;B') \]

Quantum data processing inequality
4. Discarding a system on $R$ may increase/decrease $I(R;B)$.

Eq 1. $R \rightarrow B$

Initial state: $(100) + (111)$

$E$

$S(B) - S(R;B) = 1$

Discarding $R$ (giving it to $E$)

Final state: $(100) + (111)$

$E$

$I(R' ; B) = S(B) - S(R;B) = 0$

Eq 2. $R \rightarrow B$

$I(R ; B) = S(B) - S(R;E) = 1 - 1 = 0$

$E$

$I(R' ; B) = 1 - 0 = 1$. 

$E$