Further reason to hope \( \chi^{(1)}(N) = \chi^{(n)}(N) \)


**Equivalences of additivity conjectures:**

Def. Entanglement of formation:

\[
\overline{E}_f(p) = \min_{\sum \rho_i = 1} \left\{ \sum_i \rho_i E\left( 1_{i,1} \right) \right\} S(\text{tr}_A \rho_{1,1}^{\otimes n})
\]

Average entanglement in a pure state decomposition.

\[ \rho = \sum_i \rho_i 1_{i,1} \]

Known: Entanglement dilution takes \( \sim n E\left( 1_{i,1} \right) \) ebits & \( O(\text{log} n) \) ebits to create \( n \) copies of \( 1_{i,1} \)
Intuitive protocol to make \( f \otimes n \) using ebits & obits:

Alice & Bob can use \( \approx n \) ebits to create \( 1^{4i} \) for each \( i \). Then Alice draws from \( \{p_i\} \) \( \approx d \) n times, gets \( p_{i_1}, p_{i_2}, \ldots, p_{i_n} \), tells Bob these outcomes and they place \( 1^{4i_1} \) in \( A_1B_1 \), \( 1^{4i_2} \) in \( A_2B_2 \), \ldots \( 1^{4i_n} \) in \( A_nB_n \) and forget \( p_{i_1}, \ldots, p_{i_n} \). They then share \( f \otimes n \).

Thus the name "entanglement." (That was \( \leq 1996 \)).

For a while, \( EF(f) \) was thought to be the exact asymptotic # ebits needed for any of \( f \) made.
In 2008, Hayden, Horodecki & Terhal defined the entanglement cost:

\[ E_c(p) = \lim_{n \to \infty} \frac{1}{n} \left( \text{\# bits needed to approx } p^\otimes n \right) \]

with accuracy \( \epsilon_n \) with free bits, \( \epsilon_n \to 0 \)

& proved:

\[ E_c(p) = \lim_{n \to \infty} \frac{1}{n} \left( \mathbb{E}_f (p^\otimes n) \right) \leq E_f(p) \]

if there is a better pure state decomposed over \( A_1 A_2 \ldots A_n B_1 \ldots B_m \).

For a while, everyone hoped \( E_f(p) = E_c(p) \).
In 0305035, Shor proved that the following are either all true or all false:

1. \( \forall N_1, N_2, \ X^{(1)}(N_1 \otimes N_2) = X^{(1)}(N_1) + X^{(1)}(N_2) \)

2. \( \forall f_1, f_2, \ EF(f_1 \otimes f_2) = EF(f_1) + EF(f_2) \)

3. \( \forall N_1, N_2, \ S_{\min}(N_1 \otimes N_2) = S_{\min}(N_1) + S_{\min}(N_2) \)
   (where \( S_{\min}(N) = \min_{p} S(p) \) = \( \min \) output entropy)

4. \( \forall f, a_1, b_1, \ EF_f(\gamma_{a_1,a_2} | b_1, b_2) \geq EF_f(\gamma_{a_1,b_1}) + EF_f(\gamma_{a_2,b_2}) \)