Further reason to hope $X(N) = X^*(N)$.

Equivalences of additivity conjectures:

Def. Entanglement of formation:

$$E_f(p) = \min_{\rho} \frac{\mathbb{E} \left( E(\rho) \right)}{S(\rho_{AA'BB'})}$$

Average entanglement given you choose here, $p_i, |\psi_i\rangle$: states on $AA'BB'$ decomposition.

Known: Entanglement dilution takes $\approx n E(\rho)$.

Intuitive protocol to make $\mathbb{P}^{\otimes n}$ using ebits & cbits:

Alice & Bob can use $\approx n E_f(p)$ ebits to create $1^{\otimes n}$ for each $i$. Then Alice draws from $\{p_i\}$ and in $n$ times, gets $p_1, p_2, \ldots p_n$. Tally Bob these outcomes and they place $|\psi_i\rangle$ in $A_iB_i, |\psi_2\rangle$ in $A_2B_2, \ldots |\psi_n\rangle$ in $A_nB_n$ and forget $p_1, p_2, \ldots p_n$. They then share $\mathbb{P}^{\otimes n}$.

Thus the name "entanglement formation".

For a while, $E_f(p)$ was thought to be the exact (asymptotic) # ebits needed for any copy of $\mathbb{P}$ made.

In 2008, Hayden, Linden, & Winter defined the entanglement cost:

$$E_c(p) = \lim_{n \to \infty} \frac{1}{n} \log_2 \left( \sum p_i \log_2 \left( E_f(p) \right) \right)$$

with accuracy to $1/n$ free ebits.

& proved:

$$E_c(p) = \lim_{n \to \infty} \frac{1}{n} \log_2 \left( E_f(\mathbb{P}^{\otimes n}) \right) \leq E_f(p)$$

if there is a better pure state decomposition.

For a while, everyone hoped $E_f(p) = E_c(p)$.

In 2005, Shor proved that the following are either all true or all false:

1. $\forall N_1, N_2, X^{(1)}(N_1 \otimes N_2) = X^{(1)}(N_1) + X^{(1)}(N_2)$

2. $\forall p_1, p_2, E_f(p_1 \otimes p_2) = E_f(p_1) + E_f(p_2)$

3. $\forall N_1, N_2, S_{\min}(N_1 \otimes N_2) = S_{\min}(N_1) + S_{\min}(N_2)$

(where $S_{\min}(N) = \min S(N|p) = \min S(N|p_i)$)

4. $\forall N_1, N_2, E_f(\mathbb{P}_{A_1B_1A_2B_2}) \geq E_f(\mathbb{P}_{A_1B_1}) + E_f(\mathbb{P}_{A_2B_2})$