Superdense coding: 1 ebit + 1 qbit $\geq$ 2 cbits
Teleportation: 1 ebit + 2 cbits $\geq$ 1 qbit

Resource inequality:

$\boxed{} \geq \boxed{}$

Resources on the LHS can be used to produce resource on the RHS.

Usually it means we known a protocol P for doing so.

Will see "concatenating protocols" correspond to algebraic operations of these resource inequalities.

Superdense coding: 1 ebit + 1 qbit $\geq$ 2 cbits
Teleportation: 1 ebit + 2 cbits $\geq$ 1 qbit

Can we use less in superdense coding or teleportation?

To show optimality of superdense coding, suppose

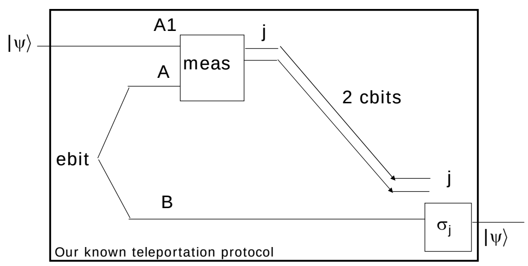
α ebit + β qbit $\geq$ 2 cbits

α, β need not be integers if we have an asymptotic (but exact) protocol and we look at the average cost.

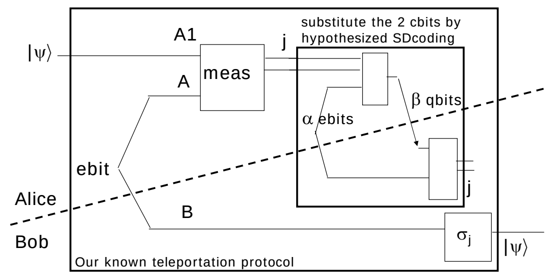
i.e. there is a (hypothesized) protocol to send 2 bits using α ebit and by sending β qubits (however complicated)

Then, we can use superdense coding to "supply" the 2 cbits needed in the teleportation protocol we already know to be working.

Using the hypothesized superdense coding protocol to "supply" the 2 cbits needed in known teleportation:



Using the hypothesized superdense coding protocol to "supply" the 2 cbits needed in known teleportation:



Net: send 1 qubit using β qbit & (1+α) ebits. Contradiction if β < 1 (for any α) (by recursive argument). So, $\beta \geq 1$.

Intriguingly, teleportation is the reason why SDcoding is optimal.

Algebraically:

Known to work:

Teleportation: 1 ebit + 2 cbits $\geq$ 1 qbit

For any superdense coding protocol:

α ebit + β qbit $\geq$ 2 cbits

math: substitute the 2nd inequality into the 1st operationally: we use the protocol (SD) giving the 2nd inequality as a BB in the protocol giving the 1st inequality. The math is valid if the composition is.

1 ebit + (α ebit + β qbit) $\geq$
1 ebit + 2 cbits $\geq$ 1 qbit

(1+α) ebits + β qbit $\geq$ 1 qbit . So, $\beta \geq 1$

Superdense coding: 1 ebit + 1 qbit $\geq$ 2 cbits
Teleportation: 1 ebit + 2 cbits $\geq$ 1 qbit

For superdense coding, suppose α ebit + β qbit $\geq$ 2 cbits

What about the entanglement cost, or α?

To analyze this, we use the fact stated last time (to be proved next class) that a quantum state in d dimensions cannot transmit more than d classical messages.

Then $\alpha + \beta \geq 2$ because the LHS is the total number of qubits in Bob's hands in the end.

So, missing ebits have to be made up for by qbits.

Since qbit $\geq$ ebit, $\alpha = \beta = 1$ optimal

Superdense coding: 1 ebit + 1 qbit $\geq$ 2 cbits
Teleportation: 1 ebit + 2 cbits $\geq$ 1 qbit

Optimality proof for teleportation is similar:

Hypothesize any other means to achieve teleportation:

γ ebit + δ cbits $\geq$ 1 qbit

To lower bound δ (the comm cost of teleportation):

Use any hypothesized teleportation to supply the 1 qbit needed in the known protocol for superdense coding.

Net resource counting for superdensing coding:
send 2 bits using δ cbits and (1+γ) ebits

A contradiction unless $\delta \geq 2$. (Run recursively to see this).

Alt: 1 ebit + (γ ebit + δ cbits)
$\geq$ 1 ebit + 1 qbit $\geq$ 2 cbits . So, $\delta \geq 2$.

Superdense coding: 1 ebit + 1 qbit $\geq$ 2 cbits
Teleportation: 1 ebit + 2 cbits $\geq$ 1 qbit

Optimality proof for teleportation is similar:

Hypothesize any other means to achieve teleportation:

γ ebit + δ cbits $\geq$ 1 qbit

To lower bound γ (the entanglement cost of teleportation):

Use teleportation to share an ebit between Alice and Bob.

So γ ebit and δ cbits generate 1 ebit.
Contradiction unless $\gamma \geq 1$.

Alt: γ ebit + δ cbits $\geq$ 1 qbit $\geq$ 1 ebit, so, $\gamma \geq 1$.

Superdense coding: 1 ebit + 1 qbit $\geq$ 2 cbits
Teleportation: 1 ebit + 2 cbits $\geq$ 1 qbit

Superdense coding and teleportation invert one another IF entanglement is free. What happens otherwise?

Note that teleportation generates 2 random bits shared between Alice and Bob. Seems like the initial ebit is degraded into these.

Superdense coding: 1 ebit + 1 qbit $\geq$ 2 cbits
Teleportation: 1 ebit + 2 cbits $\geq$ 1 qbit

Harrow 03: define cobit (coherent classical communication) as the ability to perform: $|x\rangle_A \rightarrow |x\rangle_A |x\rangle_B$ for a basis.

c.f.
qbit is the ability to perform $|x\rangle_A \rightarrow |x\rangle_B$
cbit is the ability to perform $|x\rangle_A \rightarrow |x\rangle_B |x\rangle_E$

Superdense coding: 1 ebit + 1 qbit $\geq$ 2 cbits
Teleportation: 1 ebit + 2 cbits $\geq$ 1 qbit

Harrow 03: define cobit (coherent classical communication) as the ability to perform: $|x\rangle_A \rightarrow |x\rangle_A |x\rangle_B$ for a basis.

NB: A cobit is between the ability of a qbit and an ebit:

(1) qbit $\geq$ cobit:

Protocol:
Alice applies a unitary $|x\rangle_A \rightarrow |x\rangle_A |x\rangle_{A1}$
Alice transmits A1 to Bob (using the qbit). A1 becomes B.
Net effect: $|x\rangle_A \rightarrow |x\rangle_A |x\rangle_B$ which performs a cobit.

(2) cobit $\geq$ ebit

Protocol:
Alice prepares a superposition $\sum_x |x\rangle_A$
Send A "coherently" using the cobit
Net effect: the final state $\sum_x |x\rangle_A |x\rangle_B$ is an ebit.

(3) cobit $\geq$ cbit

Superdense coding: 1 ebit + 1 qbit $\geq$ 2 cbits
Teleportation: 1 ebit + 2 cbits $\geq$ 1 qbit

Harrow 03: define cobit (coherent classical communication) as the ability to perform: $|x\rangle_A \rightarrow |x\rangle_A |x\rangle_B$ for a basis.

Why consider such an odd (if not self-ridiculed) task?

(1) Superdense coding provides more than 2 cbits!
It leaks no info to the environment. It achieves:

1 ebit + 1 qbit $\geq$ 2 cobits

Superdense coding: 1 ebit + 1 qbit $\geq$ 2 cObits
Teleportation: 1 ebit + 2 cbits $\geq$ 1 qbit

Harrow 03: define cobit (coherent classical communication) as the ability to perform: $|x\rangle_A \to |x\rangle_A\, |x\rangle_B$ for a basis.

(2) Is there an inverse?

2 cobits $\geq$ 1 qbit + 1 ebit

Not obvious, but let's try:

1 ebit + 2 cobits $\geq$ 1 qbit + 2 ebits (*)

(*) is suggestive – if there is a protocol giving (*), it's like doing teleportation using cobits and recycling 2 ebits

Teleportation:

- Alice (sender) & Bob (receiver) have initial state $|00\rangle+|11\rangle_{AB}$
- Alice's message is $|\psi\rangle_{A1} = a|0\rangle+b|1\rangle$
- Alice measures A1 & A along the basis:

  $|y_0\rangle=|00\rangle+|11\rangle$ $|y_1\rangle=|10\rangle+|01\rangle$ $|y_2\rangle=|10\rangle-|01\rangle$ $|y_3\rangle=|00\rangle-|11\rangle$

  Prior to the measurement, the state on A1 A B is
  $(a|0\rangle+b|1\rangle)_{A1}(|00\rangle+|11\rangle)_{AB} = ½ \sum_j |y_j\rangle_{A1\,A}(\sigma_j|\psi\rangle)_B$
  (just check by expanding both sides)
- If outcome is "j", postmeasurement state is $|y_j\rangle_{A1\,A}\,(\sigma_j|\psi\rangle)_B$
- Alice sends the outcome j to Bob (using 2 cbits)
- Bob applies σ_j^{-1} to B which now contains $|\psi\rangle$ regardless of j.

Teleportation: WITH COBITS

- Alice (sender) & Bob (receiver) have initial state $|00\rangle+|11\rangle_{AB}$
- Alice's message is $|\psi\rangle_{A1} = a|0\rangle+b|1\rangle$
- Alice ~~measures A1 & A along the basis:~~ transforms $|y_j\rangle$ to $|j\rangle$

  $|y_0\rangle=|00\rangle+|11\rangle$ $|y_1\rangle=|10\rangle+|01\rangle$ $|y_2\rangle=|10\rangle-|01\rangle$ $|y_3\rangle=|00\rangle-|11\rangle$

  Prior to the ~~measurement~~ unitary trsf, the state on A1 A B is
  $(a|0\rangle+b|1\rangle)_{A1}(|00\rangle+|11\rangle)_{AB} = ½ \sum_j |y_j\rangle_{A1\,A}(\sigma_j|\psi\rangle)_B$

  After the trsf, it is $½ \sum_j |j\rangle_{A1\,A}\,(\sigma_j\,|\psi\rangle)_B$
- ~~If outcome is "j", postmeasurement state is~~ $\cancel{|y_j\rangle_{A1\,A}\,(\sigma_j|\psi\rangle)_B}$
- Alice sends ~~the outcome j to Bob (using 2 cbits)~~ A1 A to Bob using 2 cobits

  resulting state: $½ \sum_j |j\rangle_{A1\,A}\,|j\rangle_{B1\,B2}\,(\sigma_j\,|\psi\rangle)_B$
- Bob applies σ_j^{-1} to B conditioned on B1 B2 being j

final state: $½ \sum_j |j\rangle_{A1\,A}\,|j\rangle_{B1\,B2}\,|\psi\rangle_B$, giving 1 qbit + 2 ebits

Superdense coding: 1 ebit + 1 qbit $\geq$ 2 cObits
Teleportation: 1 ebit + 2 cbits $\geq$ 1 qbit

Teleportation with cobits:

1 ebit + 2 cobits $\geq$ 1 qbit + 2 ebits

If we use these new ebits in later rounds of teleportation, asymptotically we get the exact inverse of SDC:

2 cobits $\geq$ 1 qbit + 1 ebit

Therefore: $\boxed{\text{2 cobits = 1 qbit + 1 ebit}}$

So, the answer to the puzzle is that, SDcoding and "teleportation with cobits" are inverses of one another. SDcoding is much better than we thought, and teleportation, naturally defined, has poorer input & output.

<u>Ensembles:</u>

Back to entropy and data compression, quantum version.

Def: Let X be a classical random var with distribution q(x).
$E=\{q(x),|\psi_x\rangle\}$ is called an *ensemble of quantum states*.
The classical rv X induces another rv which is a quantum state.

Interpretation: with prob q(x), quantum state is $|\psi_x\rangle$.

Let $\rho = \sum_x q(x)\, |\psi_x\rangle\langle\psi_x|$ be the "average" state.

e.g., let $q(1) = ½$, $|\psi_1\rangle = |0\rangle$
let $q(2) = ¼$, $|\psi_1\rangle = (|0\rangle+|1\rangle)/\sqrt{2}$
let $q(3) = ¼$, $|\psi_1\rangle = (|0\rangle-|1\rangle)/\sqrt{2}$

$$\rho = \frac{1}{2}\begin{pmatrix}1&0\\0&0\end{pmatrix} + \frac{1}{4}\frac{1}{2}\begin{pmatrix}1&1\\1&1\end{pmatrix} + \frac{1}{4}\frac{1}{2}\begin{pmatrix}1&-1\\-1&1\end{pmatrix} = \begin{pmatrix}¾&0\\0&¼\end{pmatrix}$$

$H(X) = -(½ \log ½ + ¼ \log ¼ + ¼ \log ¼) = 1.5$

<u>Ensembles:</u>

The average state: $\rho = \sum_x q(x)\, |\psi_x\rangle\langle\psi_x|$ is hermitian and trace 1, therefore it can be wrriten in terms of its eigenvalues and eigenvectors (spectral decomposition):

$\rho = \sum_v p(v)\, |e_v\rangle\langle e_v|$

von Neumann entropy:

$S(\rho)$ = [entropy of eigenvalues of ρ] = $-\mathrm{tr}\, \rho \log \rho$

For the example ensemble:

$$\rho = \frac{1}{2}\begin{pmatrix}1&0\\0&0\end{pmatrix} + \frac{1}{4}\frac{1}{2}\begin{pmatrix}1&1\\1&1\end{pmatrix} + \frac{1}{4}\frac{1}{2}\begin{pmatrix}1&-1\\-1&1\end{pmatrix} = \begin{pmatrix}¾&0\\0&¼\end{pmatrix}$$

So, $|e_1\rangle = |0\rangle$, $|e_2\rangle = |1\rangle$, $p(0) = ¾$, $p(1) = ¼$j

$S(\rho) = -¾ \log ¾ - ¼ \log ¼ = 2 - \log 3 \approx 0.8113$
but $H(X) = 1.5$

Quantum data compression (statement of result):

Suppose we make n iid draws according to $E=\{q(x),|\psi_x\rangle\}$.

i.e. with prob $q(x_1)q(x_2)\cdots q(x_n)$, state is $|\psi_{x1}\rangle|\psi_{x2}\rangle\cdots|\psi_{xn}\rangle$

We can store these n quantum states using $2^{n[S(\rho)+\varepsilon]}$ dimensions and recover them with average fidelity $\geq 1\text{-}\delta$.

NB. Need much less than $2^{n[H(q)+\varepsilon]}$ (if we're to remember $x_1\, x_2 \cdots x_n$)!!

average over the distribution of quantum states

abs(inner product) between given & recovered states

von Neumann entropy of the average state represents the space needed for compression of iid source of quantum states.

Quantum data compression (how):

Consider the spectral decomposition of average state:
$\rho = \sum_v p(v)\; |e_v\rangle\langle e_v|$

In its eigenbasis $|e_v\rangle$:
ρ is a classical random variable V with dist^n p.
$\rho^{\otimes n}$ is n iid draws of V.
Let $T_{n,\varepsilon}$ be the typical set of v^n. (Recall what that is)

For each typical sequence $v_1\, v_2 \cdots v_n$ in $T_{n,\varepsilon}$, consider the state $|e_{v1}\rangle\, |e_{v2}\rangle \cdots |e_{vn}\rangle =: |e_{v^n}\rangle$.

Together, they span typical subspace S with projector:
$\Pi_S = \sum_{v^n \in T_{n,\varepsilon}} |e_{v^n}\rangle\langle e_{v^n}|$ <-- already orthonormal

(1) $\dim S = |T_{n,\varepsilon}| \leq 2^{n(H(V)+\varepsilon)} \leq 2^{n[S(\rho)+\varepsilon]}$

(2) $\underbrace{\mathrm{Tr}(\rho^{\otimes n}\,\Pi_S)}_{\text{expand}} = \sum_{v^n \in T_{n,\varepsilon}} p(v^n) = \mathrm{prob}(T_{n,\varepsilon}) \geq 1\text{-}\delta$.

Encoding of quantum compression is simply projecting the state onto the typical space S.

(Reduces to classical data compression for an ensemble of orthogonal states. Why it works for an arbitrary ensemble is more delicate ...)

Suppose the n states to be compressed are
$|\psi_{x1}\rangle\, |\psi_{x2}\rangle \cdots |\psi_{xn}\rangle =: |\psi_{x^n}\rangle$ — the normalization inserts the prob of not outputing $|f\rangle$ automatically in the fidelity

Encoded state $= \Pi_{\mathcal{S}}\, |\psi_{x^n}\rangle$ (norm = prob of this happening)

Otherwise, output $|f\rangle$ an error symbol

Average output fidelity: $\sum_{x^n} q(x^n)\, |\underbrace{\langle\psi_{x^n}|}_{\text{input}}\; \underbrace{\Pi_S\, |\psi_{x^n}\rangle}_{\text{output}}|$

$\geq \sum_{x^n} q(x^n)\, \langle\psi_{x^n}|\, \Pi_{\mathcal{S}}\, |\psi_{x^n}\rangle$

cyclic prop trace

$= \sum_{x^n} q(x^n)\, \mathrm{Tr}\, [\, |\psi_{x^n}\rangle\langle\psi_{x^n}|\, \Pi_S]$

$= \mathrm{Tr}\, [\, \rho^{\otimes n}\, \Pi_S\,] \;\; \geq 1\text{-}\delta$ by previous slide

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More rigorous analysis using mixed state notation:

Suppose the n states to be compressed are
$|\psi_{x1}\rangle\, |\psi_{x2}\rangle \cdots |\psi_{xn}\rangle =: |\psi_{x^n}\rangle$

Encoded state $= \Pi_S\, |\psi_{x^n}\rangle\langle\psi_{x^n}|\, \Pi_S + \mathrm{Tr}[(I\text{-}\Pi_S)\, |\psi_{x^n}\rangle\langle\psi_{x^n}|\, (I\text{-}\Pi_S)]\, |f\rangle\langle f|$

Input/output fidelity:

$$\sqrt{|\langle\psi_{x^n}|\, \Pi_S\, |\psi_{x^n}\rangle\langle\psi_{x^n}|\, \Pi_S\, |\psi_{x^n}\rangle\, |}$$

For optimality of both quantum and classical data compression (i.e. if we use $2^{n(S(\rho)-\eta)}$ dimensions for any $\eta>0$, we will not recover the states), see Nielsen and Chuang.