MARKS
8
(3, 1, 2, 2)

2. A continuous random variable Y has the probability density function given at the right.

$$f(y) = \begin{cases} k(1-y^3) & ; \quad 0 < y \le 1 \\ 0 & ; \quad \text{otherwise} \end{cases}$$

   (a) Evaluate the constant k and sketch the p.d.f. of Y.
   (b) Find the mean of Y.
   (c) Find the standard deviation of Y.
   (d) Find *an expression* for the value of c so that $\Pr(-c < Y \le c) = 0.8$.

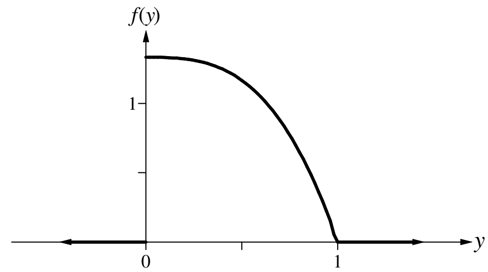
   BONUS: Find the value of c in (d) correct to 3 decimal places.

(a) Using the normalizing condition, we have: $k\int_0^1 (1-y^3)\,dy = k(y - \frac{y^4}{4})\Big|_0^1 = k[1 - \frac{1}{4} - (0-0)] = k \cdot \frac{3}{4} = 1,$

so that: $k = \frac{4}{3}$ and the p.d.f. is:

$$f(y) = \begin{cases} \frac{4}{3}(1-y^3) & ; \quad 0 < y \le 1 \\ 0 & ; \quad \text{otherwise.} \end{cases}$$

A sketch of the p.d.f. is shown at the right.



k: $\frac{4}{3} = 1.\dot{3}$ (a)

(b) $E(Y) = \int_{-\infty}^{\infty} y \cdot f(y)\,dy = \int_0^1 y \cdot \frac{4}{3}(1-y^3)\,dy = \frac{4}{3}\int_0^1 (y - y^4)\,dy$

$$= \frac{4}{3}\left(\frac{y^2}{2} - \frac{y^5}{5}\right)\Big|_0^1 = \frac{4}{3}\left(\frac{1}{2} - \frac{1}{5}\right) = \frac{4}{10} = \frac{2}{5} = 0.4.$$

Mean: 0.4 (b)

(c) $E(Y^2) = \int_{-\infty}^{\infty} y^2 \cdot f(y)\,dy = \int_0^1 y^2 \cdot \frac{4}{3}(1-y^3)\,dy = \frac{4}{3}\int_0^1 (y^2 - y^5)\,dy$

$$= \frac{4}{3}\left(\frac{y^3}{3} - \frac{y^6}{6}\right)\Big|_0^1 = \frac{4}{3}\left(\frac{1}{3} - \frac{1}{6}\right) = \frac{2}{9}.$$

Hence: $\text{s.d.}(Y) = \sqrt{E(Y^2) - [E(Y)]^2} = \sqrt{\frac{2}{9} - (\frac{2}{5})^2} = \sqrt{\frac{14}{225}} = \frac{\sqrt{14}}{15} \simeq 0.2494.$

Standard deviation: 0.2494 (c)

(d) $\Pr(-c < Y \le c) = \int_{-c}^{c} y \cdot f(y)\,dy = \frac{4}{3}\int_0^c (1-y^3)\,dy$

$$= \frac{4}{3}\left(y - \frac{y^4}{4}\right)\Big|_0^c = \frac{4}{3}\left(c - \frac{c^4}{4}\right) = 0.8;$$

$\therefore \quad c - \frac{c^4}{4} = 0.6 \qquad$ or: $\quad c^4 - 4c + 2.4 = 0.$

Expression for c: $c^4 - 4c + 2.4 = 0$ (d)

BONUS (2 marks) When $c = 0.6$, LHS = 0.1296, when $c = 0.7$, LHS = −0.1599, so the value of c lies between 0.6 and 0.7.

By trial and error; when $c = 0.641$, LHS = 0.004 823,
when $c = 0.0.642\ 6392$, LHS = 0.000 000 073,

i.e., $c = 0.643$ (correct to 3 decimal places).