Figure 5.10. PROBABILITY MODELLING: Continuous Distribution Examples

The following problems provide an opportunity to practise working with the properties of continuous probability distributions (or models). A selection of the problems will be discussed in class; *you* should try to solve the *others*, remembering that the clarity of the presentation of your solution may be as, or more, important than the final answer. Also remember to give any *final* numerical answer(s) to an appropriate number of significant figures in light of the approximate nature of *all* mathematical models of real phenomena.

Example 5.10.1: A continuous random variable Y has the probability density function given at the right.

$$f(y) = \begin{cases} \text{k}e^{1-y} & ; \quad 0 < y \le 1 \\ \text{k}/y^2 & ; \quad y > 1 \\ 0 & ; \quad \text{otherwise} \end{cases}$$

(a) Evaluate the constant k and find the c.d.f. of Y.
(b) Neatly sketch the p.d.f. of Y.
(c) Evaluate $\Pr(0.5 < Y \le 2)$ and show this probability on the graph in (b).
(d) Find the median of Y.

Example 5.10.2: A continuous random variable V has the probability density function given at the right.

$$f(v) = \begin{cases} \text{k}v(1-v)^2 & ; \quad 0 < v \le 1 \\ 0 & ; \quad \text{otherwise} \end{cases}$$

(a) Evaluate the constant k and find the c.d.f. of V.
(b) Neatly sketch the p.d.f. and c.d.f. of V.
(c) Evaluate $\Pr(0.2 < V \le 0.6)$ and show this probability on the two graphs in (b).
(d) Find the mean and standard deviation of V.

Example 5.10.3: A continuous random variable Z has the probability density function given at the right.

$$f(z) = \begin{cases} \text{k}z^2 & ; \quad 0 < z \le 1 \\ \text{k}(2-z) & ; \quad 1 < z \le 2 \\ 0 & ; \quad \text{otherwise} \end{cases}$$

(a) Evaluate the constant k and find the c.d.f. of Z.
(b) Neatly sketch the p.d.f. and c.d.f. of Z.
(c) Evaluate $\Pr(½ < Z \le 1½)$ and show this probability on the two graphs in (b).
(d) Find the mean and standard deviation of Z.

Example 5.10.4: A continuous random variable Y has the probability density function given at the right.

$$f(y) = \begin{cases} \text{k}y^{-3} & ; \quad y \ge 1 \\ 0 & ; \quad \text{otherwise} \end{cases}$$

(a) Evaluate the constant k and find the c.d.f. of Y.
(b) Neatly sketch the p.d.f. and c.d.f. of Y.
(c) Evaluate $\Pr(2 < Y \le 3)$; $\Pr(Y > 4)$; $\Pr(3.99 < Y \le 4.01)$.
(d) Find the mean and the median of Y.
(e) If possible, find the standard deviation of Y; comment briefly on what your calculations indicate.

Example 5.10.5: A continuous random variable T has the probability density function $f(t) = \text{k}(1-t^2) \quad ; \quad -1 < t \le 1$.

(a) Evaluate the constant k and find the c.d.f. of T.
(b) Neatly sketch the p.d.f. and the c.d.f. of T.
(c) Find the value of c so that $\Pr(-\text{c} < T \le \text{c}) = 0.95$.

Example 5.10.6: Find the mean and standard deviation of the random variable Y whose *c*.d.f. is given at the right.

$$F(y) = 1 - e^{-y}\left[\frac{y^4}{24} + \frac{y^3}{6} + \frac{y^2}{2} + y + 1\right] \quad ; \quad y \ge 0$$

Example 5.10.7: A continuous random variable W with the p.d.f. given at the right, where c, n and k are positive constants, is said to have a *Pareto distribution*; Pareto distributions have been used to model the distribution of incomes, where c is the subsistence wage.

$$f(w) = \begin{cases} \text{k}w^{-\text{n}-1} & ; \quad w \ge \text{c} \\ 0 & ; \quad \text{otherwise} \end{cases}$$

(a) Evaluate k (as a function of c and n) and find the c.d.f. of W.
(b) Find the median of the Pareto distribution.
(c) Find the mean and standard deviation of the Pareto distribution, including the conditions under which they are defined.

(*continued overleaf*)

Answers to the problems overleaf (but without graphs) are as follows:

Example 5.10.1: (a) $k = e^{-1}$; the c.d.f. of Y is: $F(y) = \begin{cases} 0 & ; \; y \le 0 \\ 1 - e^{-y} & ; \; 0 < y \le 1 \\ 1 - 1/ey & ; \; y > 1 \end{cases}$

(b)

(c) $e^{-\frac{1}{2}} - \frac{1}{2e} \simeq 0.4226$.

(d) median $= \ln 2 \simeq 0.6931$.

Example 5.10.2: (a) $k = 12$; the c.d.f. of V is: $F(v) = \begin{cases} 0 & ; \; v \le 0 \\ v^2(6 - 8v + 3v^2) & ; \; 0 < v \le 1 \\ 1 & ; \; v > 1 \end{cases}$

(b)

(c) 0.64.

(d) $E(V) = 0.4$; $s.d.(V) = 0.2$.

Example 5.10.3: (a) $k = \frac{6}{5}$; the c.d.f. of Z is: $F(z) = \begin{cases} 0 & ; \; z \le 0 \\ \frac{2}{5}z^3 & ; \; 0 < z \le 1 \\ \frac{1}{5}(-7 + 12z - 3z^2) & ; \; 1 < z \le 2 \\ 1 & ; \; z > 2 \end{cases}$

(b)

(c) $\frac{4}{5}$

(d) $E(Z) = 1.1$; $s.d.(Z) = \sqrt{13}/10 \simeq 0.3606$.

Example 5.10.4: (a) $k = 2$; the c.d.f. of Y is: $F(y) = \begin{cases} 0 & ; \; y < 1 \\ 1 - 1/y^2 & ; \; y \ge 1 \end{cases}$

(b)

(c) $\frac{5}{36} = 0.13\dot{8}$; $\frac{1}{16}$; 0.000 625 (007 813).

(d) $E(Y) = 2$; median $= \sqrt{2} \simeq 1.4142$.

(e) The standard deviation is undefined because $E(Y^2)$ is infinite; this occurs because .....

Example 5.10.5: (a) $k = \frac{3}{4}$; the c.d.f. of T is: $F(t) = \begin{cases} 0 & ; \; t \le -1 \\ \frac{1}{4}[2 + t(3 - t^2)] & ; \; -1 < t \le 1. \\ 1 & ; \; t > 1 \end{cases}$

(b)

(c) $c^3 - 3c + 1.9 = 0$, so that: $c \simeq 0.811(401\,352)$.

Example 5.10.6: $E(Y) = 5$; $s.d.(Y) = \sqrt{5} \simeq 2.2361$.

Example 5.10.7: (a) $k = nc^n$; the c.d.f. of W is: $F(w) = \begin{cases} 0 & ; \; w < c \\ 1 - \left(\frac{c}{w}\right)^n & ; \; w \ge c \end{cases}$

(b) median $= c/\sqrt[n]{2}$.

(c) $E(W) = \dfrac{nc}{n-1}$ (defined for $n > 1$);

$s.d.(W) = \dfrac{c}{n-1}\sqrt{\dfrac{n}{n-2}}$ (defined for $n > 2$).