Robustness in the Optimization of Risk Measures

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Agenda

1. Risk measures
2. Classic statistical robustness
3. Robustness in optimization
4. VaR and ES in representative optimization problems
5. Is distributionally robust optimization robust?
6. Conclusion

Based on joint work with Paul Embrechts (Zurich) and Alexander Schied (Waterloo)

Risk Measures

A risk measure $\rho : \mathcal{X} \to \overline{\mathbb{R}} = (-\infty, \infty]$

- Risks are modelled by random losses in a specified period
  - e.g. 10d in Basel III & IV market risk
- $\mathcal{X}$ is a convex cone of rvs in some probability space $(\Omega, \mathcal{F}, \mathbb{P})$

Roles of risk measures

- regulatory capital calculation $\leftarrow$ our main interpretation
- management, optimization and decision making
- performance analysis and capital allocation
- risk pricing

General Question

Question

What is a "good" risk measure for regulatory capital calculation?

- Regulator's and firm manager's perspectives can be different or even conflicting
  - well-being of the society versus interest of the shareholders
  - systemic risk in an economy versus risk of a single firm

Value-at-Risk and Expected Shortfall

Value-at-Risk (VaR) at level $p \in (0,1)$

$\mathrm{VaR}_p : L^0 \to \mathbb{R}$,

$$\mathrm{VaR}_p(X) = F_X^{-1}(p) = \inf\{x \in \mathbb{R} : \mathbb{P}(X \le x) \ge p\}.$$

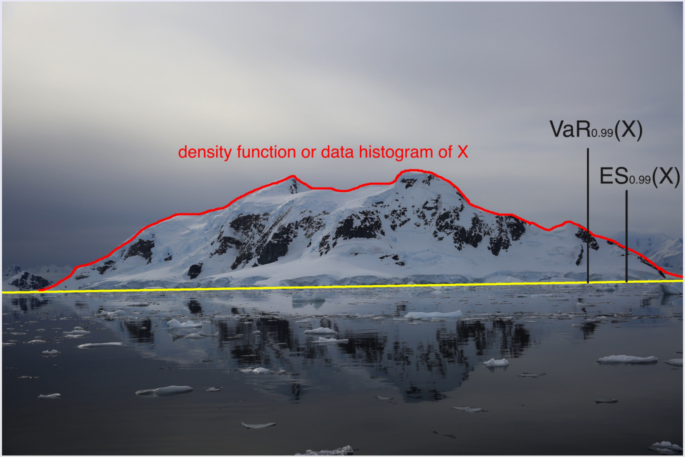
Expected Shortfall (ES/TVaR/CVaR/AVaR) at level $p \in (0,1)$

$\mathrm{ES}_p : L^0 \to \overline{\mathbb{R}}$,

$$\mathrm{ES}_p(X) = \frac{1}{1-p}\int_p^1 \mathrm{VaR}_q(X)\mathrm{d}q \underset{(F_X \text{ cont.})}{=} \mathbb{E}\left[X | X > \mathrm{VaR}_p(X)\right].$$

F_X above is the distribution function of X.

Value-at-Risk and Expected Shortfall



Value-at-Risk and Expected Shortfall

The ongoing co-existence of VaR and ES:

- Basel IV - both
- Solvency II - VaR
- Swiss Solvency Test - ES

Academic Inputs

- ES is generally advocated by academia for desirable properties in the past two decades; in particular,
  - subadditivity or coherence (Artzner-Delbaen-Eber-Heath'99)
  - convex optimization properties (Rockafellar-Uryasev'00)
- Some other examples of impact from academic research
  - Gneiting'11: backtesting ES is unclear, whereas backtesting VaR is straightforward
  - Cont-Deguest-Scandolo'10: ES is not robust, whereas VaR is

VaR versus ES

BCBS Consultative Document, May 2012, Page 41, *Question 8*:

"*What are the likely constraints with moving from VaR to ES, including any challenges in delivering robust backtesting, and how might these be best overcome?*"

VaR versus ES

Features/Risk measure	VaR	Tail-VaR
Frequency captured?	Yes	Yes
Severity captured?	No	Yes
Sub-additive?	Not always	Always
Diversification captured?	Issues	Yes
Back-testing?	Straight-forward	Issues
Estimation?	Feasible	Issues with data limitation
Model uncertainty?	Sensitive to aggregation	Sensitive to tail modelling
Robustness I (with respect to "Lévy metric[33]")?	Almost, only minor issues	No
Robustness II (with respect to "Wasserstein metric[34]")?	Yes	Yes

Table copied from IAIS Consultation Document Dec 2014, page 42

Progress

Model Uncertainty

VaR and ES are law-based (thus statistical risk functionals):
$\rho(X) = \rho(Y)$ if $X \stackrel{\mathrm{d}}{=}_{\mathbb{P}} Y$ (equal in distribution under $\mathbb{P}$)

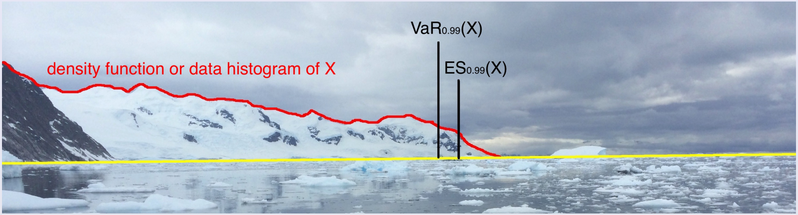
- The calculation requires knowledge of the distribution of a risk
- This may never be the exact case: model uncertainty
  - statistical error
  - computational error
  - modeling error
  - conceptual error
- Models are at most "approximately correct" $\Rightarrow$ robustness!

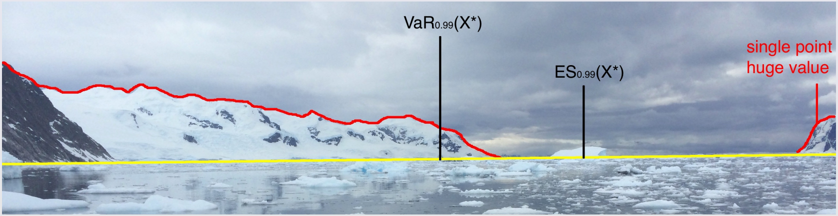
Robust Statistics

Statistical robustness addresses the question of "what if the data is compromised with small error?"

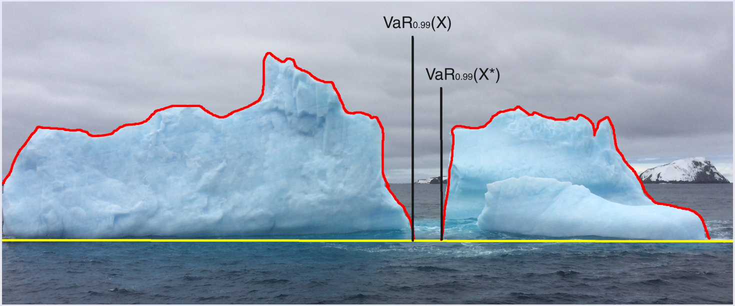
- Originally robustness is defined on estimators (estimation procedures)
- Would the estimation be ruined if the underlying model is compromised?
  - e.g. an outlier is added to the sample

VaR and ES Robustness





VaR and ES Robustness



- Non-robustness of VaR_p only happens if the quantile has a gap at p
- Is this situation relevant for risk management practice?
  - one must be very unlucky to hit precisely where it has a gap ...

Robust Statistics

Classic qualitative robustness:

- Hampel'71: the robustness of a consistent estimator of T is equivalent to the continuity of T with respect to underlying distributions (both with respect to the same metric)
- When we talk about the robustness of a statistical functional, (Huber-Hampel's) robustness typically refers to continuity with respect to some metric.
- (Pseudo-)metrics: $\pi^q = L^q$ $(q \geq 1)$, $\pi^\infty = L^\infty$, $\pi^W =$ Lévy, ...

General reference: Huber-Ronchetti'07

Robustness of Risk Measures

Consider the continuity of $\rho : \mathcal{X} \to \mathbb{R}$.

- A strong sense of continuity is w.r.t. weak convergence.
  - $X_n \to X$ in distribution $\Rightarrow \rho(X_n) \to \rho(X)$.
- Quite restrictive
- Practitioners like weak convergence (e.g. estimation, simulation)

Robustness of Risk Measures

- With respect to weak convergence $p \in (0,1)$:
  - VaR_p is continuous at distributions whose quantile is continuous at p. VaR_p is argued as being almost robust.
  - ES_p is not continuous for any $\mathcal{X} \supset L^\infty$
- ES_p is continuous w.r.t. some other (stronger) metric, e.g. π^q (or the Wasserstein-L^q metric)

Range-Value-at-Risk (RVaR)

A two-parameter family of risk measures, for $\alpha, \beta > 0$, $\alpha + \beta < 1$,

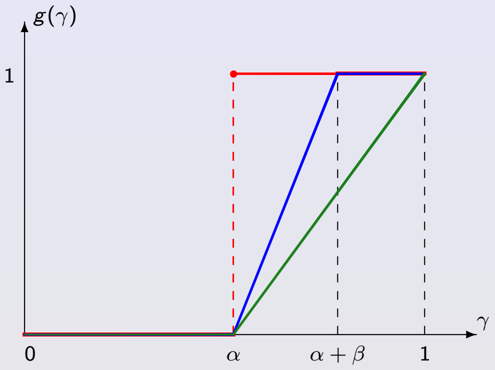
$$\mathrm{RVaR}_{\alpha,\beta}(X) = \frac{1}{\beta} \int_{\alpha}^{\alpha+\beta} \mathrm{VaR}_{\gamma}(X) \mathrm{d}\gamma, \quad X \in \mathcal{X}.$$

- RVaR bridges the gap between VaR and ES (limiting cases).
- RVaR is continuous w.r.t. weak convergence
- RVaR is not convex or coherent
- Practically:

$$\mathrm{RVaR}_{\alpha,\beta}(X) \underset{(F_X \text{ cont.})}{=} \mathbb{E}[X | \mathrm{VaR}_{\alpha}(X) < X \leq \mathrm{VaR}_{\alpha+\beta}(X)].$$

First proposed by Cont-Deguest-Scandolo'10; name in W.-Bignozzi-Tsanakas'15

Range-Value-at-Risk (RVaR)



Distortion functions of VaR_α (red), ES_α (green) and $\mathrm{RVaR}_{\alpha,\beta}$ (blue)

in the form of $\int_0^1 \mathrm{VaR}_\gamma(X)\mathrm{d}g(\gamma)$

Classic Robustness

The general perception of robustness, from worst to best:

$$\mathrm{ES} \prec \mathrm{VaR} \prec \mathrm{RVaR}$$

From weak to strong:

- Continuity w.r.t. π^∞: all monetary risk measures
- Continuity w.r.t. π^q, $q \geq 1$: finite convex risk measures on L^q, e.g. ES_p
- Continuity w.r.t. weak/a.s./P convergence: e.g. $\mathrm{RVaR}_{\alpha,\beta}$, VaR_p (almost); no convex risk measure satisfies this

Bäuerle-Müller'06, Cont-Deguest-Scandolo'10, Kou-Peng-Heyde'13

Robustness of Risk Measures

Is robustness w.r.t. weak convergence necessarily a good thing?

- Toy example.
  - Let $X_n = n^2 \mathbb{1}_{\{U \le 1/n\}}$ for some U[0,1] random variable U (e.g. a credit default risk). Clearly $X_n \to 0$ a.s. but X_n is getting more "dangerous" in many senses. If ρ preserves weak convergence, then

$$\rho(X_n) \to \rho(0) \quad (= 0 \quad \text{typically}).$$

  - $\mathrm{VaR}_{0.999}(X_{10000}) = 0$
  - $\mathrm{ES}_{0.999}(X_{10000}) = 10^7$
- May be reasonable for internal management; not so much for regulation.

One-in-ten-thousand Event

On the other hand,

- the 1/10,000-event-type risks are very difficult to capture statistically (accuracy is impossible)

UK House of Lords/House of Commons, June 12, 2013, Output of a "stress test" exercise, from HBOS:

"*We actually got an external advisor [to assess how frequently a particular event might happen] and they came out with one in 100,000 years and we said "no", and I think we submitted one in 10,000 years. But that was a year and a half before it happened. It doesn't mean to say it was wrong: it was just unfortunate that the 10,000th year was so near.*"

Progress

Motivation

- So far, VaR and ES are applied to the same financial position.
- The regulatory choice of ρ creates certain incentives, effective before ρ is applied to assess risks.
- Once a specific ρ has been chosen, portfolios will be optimized with respect to ρ (at least to some extend).
- In reality, VaR and ES will not be applied to the same position.

One cannot decouple the technical properties of a risk measure from the incentives it creates.

The Optimization Problem

General setup

- $\mathcal{G}_n = \{$measurable functions from $\mathbb{R}^n$ to $\mathbb{R}\}$
- $X \in (L^0)^n$ is an economic vector, representing all random sources
- $\mathcal{G} \subset \mathcal{G}_n$ is a decision set
- $g(X)$ for $g \in \mathcal{G}$ represents a risky position of an investor
- ρ is an objective functional mapping $\{g(X) : g \in \mathcal{G}\}$ to $\overline{\mathbb{R}}$

"The optimization problem":

to minimize $\rho(g(X))$ over $g \in \mathcal{G}$

The Optimization Problem

Denote (possibly empty)

$$\mathcal{G}^*(X,\rho) = \left\{ g \in \mathcal{G} : \rho(g(X)) = \inf_{h \in \mathcal{G}} \rho(h(X)) \right\}.$$

We call

- $g^* \in \mathcal{G}^*(X,\rho)$ an optimizing function
- $g^*(X)$ an optimized position

Uncertainty in Optimization

- The optimization problem is subject to model uncertainty
- Let $\mathcal{Z}$ be a set of possible economic vectors including X
  - $\mathcal{Z}$: the set of alternative models
  - e.g. a parametric family of models (parameter uncertainty)
- The true economic vector $Z \in \mathcal{Z}$ is likely different from the perceived economic vector X
  - X: best-of-knowledge model
  - Z: true model (unknowable)
- $g_X \in \mathcal{G}^*(X, \rho)$ is a best-of-knowledge decision
  - true position $g_X(Z)$
  - perceived position $g_X(X)$

Uncertainty in Optimization

We are interested in the insolvency gap

$$\underbrace{\rho(g_X(Z))}_{\text{true risk}} - \underbrace{\rho(g_X(X))}_{\text{perceived risk}}$$

not the optimality gap

$$\underbrace{\rho(g_Z(Z))}_{\text{true optimum}} - \underbrace{\rho(g_X(Z))}_{\text{true risk}}$$

or the difference between optima

$$\underbrace{\rho(g_Z(Z))}_{\text{true optimum}} - \underbrace{\rho(g_X(X))}_{\text{perceived optimum}}$$

Uncertainty in Optimization

- If the modeling has good quality, Z and X are close to each other according to some metric π
- $\rho(g_X(Z))$ should be close to $\rho(g_X(X))$ to make sense of the optimizing function g_X $\Rightarrow$ some continuity of the mapping $Z \mapsto \rho(g_X(Z))$ at $Z = X$
- We call $(\mathcal{G}, \mathcal{Z}, \pi)$ an uncertainty triplet if $\mathcal{G} \subset \mathcal{G}_n$ and $(\mathcal{Z}, \pi)$ is a pseudo-metric space of n-random vectors.
- ρ is compatible if its domain contains $\mathcal{G}(\mathcal{Z})$ and $\rho(g(Y)) = \rho(g(Z))$ for all $g \in \mathcal{G}$ and $Y, Z \in \mathcal{Z}$ with $\pi(Y, Z) = 0$.

Robustness in Optimization

Definition 1

A compatible objective functional ρ is robust at $X \in \mathcal{Z}$ relative to the uncertainty triplet $(\mathcal{G}, \mathcal{Z}, \pi)$ if there exists $g \in \mathcal{G}^*(X, \rho)$ such that the function $Z \mapsto \rho(g(Z))$ is π-continuous at $Z = X$.

- Robustness is a joint property of the tuple $(\rho, X, \mathcal{G}, \mathcal{Z}, \pi)$
- Only a π-neighbourhood of X in $\mathcal{Z}$ matters

Robustness in Optimization

Remarks.

- If ρ is robust at X relative to $(\mathcal{G}, \mathcal{Z}, \pi)$, then it also holds
  - relative to $(\mathcal{G}, \mathcal{Z}', \pi)$ if $X \in \mathcal{Z}' \subset \mathcal{Z}$;
  - relative to $(\mathcal{G}, \mathcal{Z}, \pi')$ if π' is stronger than π
- If $\mathcal{G}^*(X, \rho) = \varnothing$, then ρ is not robust at X
-
  - One can use topologies instead of metrics
  - One can consider uncertainty on the set of probability measures instead of on the set of random vectors
  - One can require the continuity for all $g \in \mathcal{G}^*(X, \rho)$ instead of that for some g.

Progress

1. Risk measures
2. Classic statistical robustness
3. Robustness in optimization
4. VaR and ES in representative optimization problems
5. Is distributionally robust optimization robust?
6. Conclusion

Representative Optimization Problems

Representative optimization problems.

- $n = 1$ and $X \geq 0$ is a random loss
- The pricing density $\gamma = \gamma(X)$ is a measurable function of X
  - $\gamma > 0$, $\mathbb{E}[\gamma] = 1$ and $\mathbb{E}[\gamma X] < \infty$
- The budget constraint is $\mathbb{E}[\gamma g(X)] \geq x_0$
- Problems: to minimize $\rho(g(X))$ over $g \in \mathcal{G}$ for some $\mathcal{G} \subset \mathcal{G}_1$ in three settings $\mathcal{G} = \mathcal{G}_{\rm cm}, \mathcal{G}_{\rm ns}, \mathcal{G}_{\rm bd}$

Representative Optimization Problems

(a) Complete market:

$$\mathcal{G}_{\mathrm{cm}} = \{g \in \mathcal{G}_1 : \mathbb{E}[\gamma g(X)] \geq x_0\}.$$

(b) No short-selling or over-hedging constraint:

$$\mathcal{G}_{\mathrm{ns}} = \{g \in \mathcal{G}_1 : \mathbb{E}[\gamma g(X)] \geq x_0,\ 0 \leq g(X) \leq X\}.$$

Assume $0 \leq x_0 < \mathbb{E}[\gamma X]$ to avoid triviality.

(c) Bounded constraint: for some $m > 0$,

$$\mathcal{G}_{\mathrm{bd}} = \{g \in \mathcal{G}_1 : \mathbb{E}[\gamma g(X)] \geq x_0,\ 0 \leq g(X) \leq m\}.$$

Assume $0 \leq x_0 < m$ to avoid triviality.

Representative Optimization Problems

Remark.

- Problem (c) is not a special case of Problem (b) as X in (b) is both the constraint and the source of randomness

For (a)-(c), assume

- The distribution function of X is continuous and strictly increasing on $(\text{ess-inf} X, \text{ess-sup} X)$.
- $(\mathcal{Z}, \pi)$ is one of the classic choices (L^q, π^q) for $q \in [1, \infty]$ and (L^0, π^W), and $X \in \mathcal{Z}$.

We focus on VaR_p and ES_p for $p \in (0, 1)$.

Problem (c) for distortion risk measures: He-Zhou'11

Robustness in the Optimization of VaR

Let

$$q = \inf \left\{ \mathrm{VaR}_p(g(X)) : g \in \mathcal{G}_{\mathrm{ns}} \right\},$$

$$q' = \inf \left\{ \mathrm{VaR}_p(g(X)) : g \in \mathcal{G}_{\mathrm{bd}} \right\}.$$

Assumption 1

$q > 0$ *and* $\mathbb{P}((X-q)\gamma \le \mathrm{VaR}_p((X-q)\gamma)) = p$.

Assumption 2

$q' > 0$ *and* $\mathbb{P}(\gamma \le \mathrm{VaR}_p(\gamma)) = p$.

- $q, q' > 0$ means the optimization does not result in zero risk
- Assumptions 1-2 are very weak

Solutions to the Representative Problems for VaR_p

Proposition 1 (VaR_p, Problem (c))

Let U be a uniform transform of γ on the probability space $(\Omega, \sigma(X), \mathbb{P})$.

(i) $q' = 0$ *if and only if* $\mathrm{mES}_p(\gamma) \geq \frac{x_0}{1-p}$.

(ii) *If* $q' = 0$*, a solution of Problem (c) is given by*

$$g^*(X) = m\mathbb{1}_{\{U>p\}}.$$

(iii) *If* $q' > 0$*, any solution to Problem (c) has the form*

$$g^*(X) = m\mathbb{1}_{\{U>p\}} + q'\mathbb{1}_{\{U\leq p\}}, \text{ a.s.}$$

Robustness in the Optimization of VaR

Theorem 1

For $p \in (0,1)$ and $X \in \mathcal{Z}$,

(i) VaR_p *is not robust relative to* $(\mathcal{G}_{\mathrm{cm}}, \mathcal{Z}, \pi)$*;*

(ii) *under Assumption 1,* VaR_p *is not robust at* X *relative to* $(\mathcal{G}_{\mathrm{ns}}, \mathcal{Z}, \pi)$*;*

(iii) *under Assumption 2,* VaR_p *is not robust at* X *relative to* $(\mathcal{G}_{\mathrm{bd}}, \mathcal{Z}, \pi)$*.*

- Robustness of VaR in optimization is very bad

Robustness in the Optimization of ES

Assumption 3

$\text{ess-sup}\gamma \le \frac{1}{1-p}$.

- Assumption 3 may be interpreted as a no-arbitrage condition for a market with ES participants

Assumption 4

Either γ is a constant, or γ is a continuous function and $\gamma(X)$ is continuously distributed.

- Assumption 4 is commonly satisfied

Robustness in the Optimization of ES

Theorem 2

For $p \in (0,1)$ and $X \in \mathcal{Z}$,

(i) *under Assumption 3, ES_p is robust at X relative to $(\mathcal{G}_{\mathrm{cm}}, \mathcal{Z}, \pi)$;*

(ii) *under Assumption 4, ES_p is robust at X relative to $(\mathcal{G}_{\mathrm{ns}}, \mathcal{Z}, \pi)$ for $(\mathcal{Z}, \pi) = (L^q, \pi^q)$, $q \in [1, \infty]$;*

(iii) *under Assumption 4, ES_p is robust at X relative to $(\mathcal{G}_{\mathrm{bd}}, \mathcal{Z}, \pi)$.*

- Robustness of ES in optimization is quite good

Robustness in Optimization for VaR and ES

On robustness in optimization:

$$\mathrm{VaR} \lll \mathrm{ES} \quad (\mathrm{RVaR}/\mathrm{ES} \text{ not easy to compare})$$

Observations.

- The discontinuity in $Z \mapsto g^*(Z)$ comes from the fact that optimizing VaR is "too greedy": always ignores tail risk, and hoping the probability of the tail risk is correctly modelled.
- None of the two values

$$\mathrm{VaR}_p(g^*(X)) \text{ and } \mathrm{VaR}_p(g^*(Z))$$

is a rational measure of the "optimized" risk.

Robustness in Optimization for VaR and ES

Is risk positions of type g^* realistic?

"Starting in 2006, the CDO group at UBS noticed that their risk-management systems treated AAA securities as essentially riskless even though they yielded a premium (the proverbial free lunch). So they decided to hold onto them rather than sell them. "

- From Feb 06 to Sep 07, UBS increased investment in AAA-rated CDOs by more than 10 times; many large banks did the same.
  - Take a risk of big loss with small probability
  - Treat it as free money - profit
  - Model uncertainty?

---

quoted from Acharya-Cooley-Richardson-Walter'10

Progress

Is Distributionally Robust Optimization Robust?

Distributionally robust optimization, for $\epsilon > 0$:

$$\text{to minimize:} \quad \sup_{\pi(Y,X)\le\epsilon} \rho(g(Y)) \quad \text{subject to } g \in \mathcal{G}.$$

- $\mathcal{G}^*(X, \rho, \epsilon)$: the set of functions $g \in \mathcal{G}$ solving this problem
- $\epsilon = 0$ leads to $\mathcal{G}^*(X, \rho, 0) = \mathcal{G}^*(X, \rho)$, the original setting
- ρ is robust for the ϵ-problem if there exists $g \in \mathcal{G}^*(X, \rho, \epsilon)$ such that $Z \mapsto \rho(g(Z))$ is π-continuous at $Z = X$
- This type of problems is hard to solve and we focus on VaR_p for Problem (c): $(\mathcal{G}, \mathcal{Z}, \pi) = (\mathcal{G}_{\mathrm{bd}}, L^\infty, \pi^\infty)$.

---

e.g. Natarajan-Pachamanova-Sim'08, Zhu-Fukushima'09,

Is Distributionally Robust Optimization Robust?

The problem: to minimize

$$\sup_{\pi^\infty(Y,X)\le\epsilon} \mathrm{VaR}_p(g(Y)) \text{ subject to } g\in\mathcal{G}_{\mathrm{bd}},$$

where $\mathcal{G}_{\mathrm{bd}}=\{g\in\mathcal{G}_1:\mathbb{E}[\gamma g(X)]\ge x_0,\ 0\le g(X)\le m\}$. Let

$$q_\epsilon=\inf\left\{\sup_{\pi^\infty(Y,X)\le\epsilon}\mathrm{VaR}_p(g(Y)):g\in\mathcal{G}_{\mathrm{bd}}\right\}.$$

Assumption 5

$q_\epsilon>0$, $1/2\le p<1$, X *has a decreasing density on* $(\operatorname{ess-inf}X,\operatorname{ess-sup}X)$ *and* γ *is an increasing function of* X.

Is Distributionally Robust Optimization Robust?

Proposition 2

Under Assumption 5, the above problem admits a solution of the form

$$g^*(x) = m\mathbb{1}_{\{x>c+\epsilon\}} + q_\epsilon \mathbb{1}_{\{x\le c+\epsilon\}}, \ x \in \mathbb{R}, \quad \textit{where } c = \mathrm{VaR}_p(X).$$

- $Z \mapsto \mathrm{VaR}_p(g^*(Z))$ is π^∞-continuous at $Z = X$
- VaR_p is robust for the ϵ-problem
- The ϵ-modification improves the robustness of VaR
- We still get the big-loss-small-probability type of optimizer

Progress

Conclusion

Some conclusions on robustness

- Classic notion
  - $\mathrm{ES} \prec \mathrm{VaR} \prec \mathrm{RVaR}$
  - However this robustness may not be desirable
- If we take optimization into account
  - $\mathrm{VaR} \lll \mathrm{ES}$ in optimization
  - The rationality of optimizing VaR under model uncertainty is questionable
- Some other perspectives
  - $\mathrm{VaR} \prec \mathrm{ES} \prec \mathrm{RVaR}$ in risk aggregation
  - $\mathrm{VaR} \lll \mathrm{ES} \prec \mathrm{RVaR}$ in risk sharing

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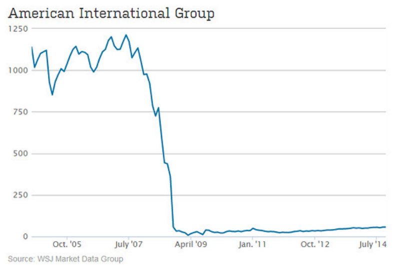
Embrechts-Wang-W.'15, Krätschmer-Schied-Zähle'17, Embrechts-Liu-W.'18

Other Questions

Many other questions ...

- other risk measures
- other optimization problems
- utility maximization problems
- risk measures as constraints instead of objectives

AIG



CEO of AIG Financial Products, August 2007:

"*It is hard for us, without being flippant, to even see a scenario within any kind of realm of reason that would see us losing one dollar in any of those transactions.*"

- AIGFP sold protection on super-senior tranches of CDOs
- $180 billion bailout from the federal government in September 2008

Thank You

VaR   Real danger

This paper is available on SSRN (3254587) and arXiv (1809.09268)

Solutions to the Representative Problems

Proposition 3 (VaR_p, Problem (a))

$\inf\{\mathrm{VaR}_p(g(X)) : g \in \mathcal{G}_{\mathrm{cm}}\} = -\infty$. *Hence, Problem (a) admits no solution.*

Solutions to the Representative Problems for VaR_p

Proposition 4 (VaR_p, Problem (b))

Let U be a uniform transform of $(X-q)\gamma$ on the probability space $(\Omega, \sigma(X), \mathbb{P})$.

(i) $q = 0$ *if and only if* $\mathrm{ES}_p(\gamma X) \geq \frac{x_0}{1-p}$.

(ii) *If $q = 0$, a solution of Problem (b) is given by*

$$g^*(X) = X\mathbb{1}_{\{U>p\}}.$$

(iii) *If $q > 0$, any solution to Problem (b) has the form*

$$g^*(X) = X\mathbb{1}_{\{U>p\}} + (X \wedge q)\mathbb{1}_{\{U\leq p\}}, \text{ a.s.}$$

Solutions to the Representative Problems for VaR_p

Proposition 5 (VaR_p, Problem (c))

Let U be a uniform transform of γ on the probability space $(\Omega, \sigma(X), \mathbb{P})$.

(i) $q' = 0$ *if and only if* $\mathrm{mES}_p(\gamma) \geq \frac{x_0}{1-p}$.

(ii) *If* $q' = 0$, *a solution of Problem (c) is given by*

$$g^*(X) = m\mathbb{1}_{\{U>p\}}.$$

(iii) *If* $q' > 0$, *any solution to Problem (c) has the form*

$$g^*(X) = m\mathbb{1}_{\{U>p\}} + q'\mathbb{1}_{\{U\leq p\}}, \text{ a.s.}$$

Solutions to the Representative Problems for ES_p

Proposition 6 (ES_p, Problem (a))

Problem (a) admits a solution if and only if Assumption 3 holds, and if Assumption 3 holds, a solution is given by $g^*(\cdot) = x_0$.

Solutions to the Representative Problems for ES_p

Proposition 7 (ES_p, Problem (b))

There exist constants $c > 0$, $r \geq 0$, and $\lambda \in [0,1]$ such that the function g^*, for $x \in \mathbb{R}$,

$$g^*(x) = x\mathbb{1}_{\{\gamma(x)>c\}} + (x \wedge r)\mathbb{1}_{\{\gamma(x)<c\}} + ((1-\lambda)x + \lambda(x \wedge r))\mathbb{1}_{\{\gamma(x)=c\}},$$

solves Problem (b). Moreover, r is a p-quantile of $g^*(X)$.

Solutions to the Representative Problems for ES_p

Proposition 8 (ES_p, Problem (c))

There exist constants $c > 0$, $r \in [0, m]$, and $\lambda \in [r, m]$ such that the function g^*, for $x \in \mathbb{R}$,

$$g^*(x) = m\mathbb{1}_{\{\gamma(x)>c\}} + r\mathbb{1}_{\{\gamma(x)<c\}} + \lambda\mathbb{1}_{\{\gamma(x)=c\}},$$

solves Problem (c). Moreover, r is a p-quantile of $g^*(X)$.

Industry Perspectives

From the **International Association of Insurance Supervisors**:

- Document (version June 2015)
  Compiled Responses to ICS Consultation 17 Dec 2014 - 16 Feb 2015

In summary

- Responses from insurance organizations and companies in the world.
- 49 responses are public
- 34 commented on Q42: VaR versus ES (TVaR)

Industry Perspectives

- 5 responses are supportive about ES:
  - Canadian Institute of Actuaries, CA
  - Liberty Mutual Insurance Group, US
  - National Association of Insurance Commissioners, US
  - Nematrian Limited, UK
  - Swiss Reinsurance Company, CH
- Some are indecisive; most favour VaR.

Regulator and firms may have different views

Discussion

Major reasons to favour VaR from the insurance industry (IAIS report June 2015)

- Implementation of ES is expensive (staff, software, capital)
- ES does not exist for certain heavy-tailed risks
- ES is more costly on distributional information, data and simulation
- ES has trouble with a change of currency