LECTURES ON THE KADISON-SINGER PROBLEM FALL 2007

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Abstract. These notes cover various topics in C*-algebras and functional analysis related to the Kadison-Singer Problem[?]. They are intended for an audience that is familiar with some of the basic results in the theory of Banach and C*-algebras, but have sufficient references that they can be read with somewhat less preparation.

These notes are based on a course taught in the Fall 2007 at the University of Houston. I am grateful to the students of that course, Ali S. Kavruk, Sneh Lata, Meghna Mittal, Mrinal Raghupathi, Preeti Singh and Blerina Xhabli, for taking notes and helping with the initial texing.

1. Introduction

The field of C*-algebras has become so broad and diverse that it is truly a heroic effort to attempt to learn the entire field without a particular goal in mind. On the other hand this is rarely the way that active research mathematicians learn a new field. Generally, we learn the parts of a field that are revelant to the problem that we are focused on, while, hopefully, learning enough about each subarea that we are comfortable with that area and at least know where to look if we need further details.

I've designed these lectures around the material that one needs to know in order to study the still unsolved Kadison-Singer problem[?]. Of course, we must assume that the reader knows something, so we will assume that the reader is familiar with the material in Chapters VII and VIII of Conway's book [?] or Chapter I of Davidson's book [?]. Although we will review some of these ideas below, but not in a comprehensive(or perhaps comprehensible!) manner.

The goal of these lectures isn't necessarily to prepare the student to do research on the Kadison-Singer problem, although these notes should prepare you for that. But instead they use the Kadison-Singer problem as a point of departure to introduce an array of topics in C*-algebras. This approach wouldn't work for every unsolved problem in C*-algebras, but, fortunately, the Kadison-Singer problem is a problem that seems to be so fundamental(although at first it doesn't look that way at all!) that it impinges on many areas.

So without further ado, we state the problem and outline our lectures:

Date: December 10, 2007.

2000 *Mathematics Subject Classification.* Primary 46L15; Secondary 47L25.

The Kadison-Singer Problem: Let $\mathcal{H}$ be a separable Hilbert space and let $\mathcal{D} \subseteq B(\mathcal{H})$ be a discrete MASA. Does every pure state on $\mathcal{D}$ extend to a unique pure state on $B(\mathcal{H})$?

Clearly, to understand precisely what the problem is asking we will need to first understand what are pure states and what is a discrete MASA. Here is the rough plan of our lectures:

I. Some C*-algebra basics,
II. States and pure states on C*-algebras,
III. Discrete and continuous MASA's,
IV. The Stone-Cech compactification,
V. Ultrafilters,
VI. Ultrafilters and $\beta\mathbb{N}$,
VII. Anderson's paving results,
VIII. Other paving results,
IX. Introduction to frames,
X. Frames and paving,
XI. Introduction to groups actions and crossed-products,
XII. Crossed-products and Kadison-Singer,
XIII. Dynamical systems and βG,
XIV. Algebra in βG.

2. Some C*-algebra basics

Let $\mathcal{V}$ be a complex vector space. By an **involution** on $\mathcal{V}$ we mean a map, $^* : \mathcal{V} \to \mathcal{V}$, satisfying:

- $(v^*)^* = v$, for every $v \in \mathcal{V}$,
- $(v + w)^* = v^* + w^*$, for every $v, w \in \mathcal{V}$,
- $(\lambda v)^* = \bar{\lambda} v^*$, for every $\lambda \in \mathbb{C}$ and $v \in \mathcal{V}$.

A complex vector with an involution is often referred to as a ***-vector space.**

If $\mathcal{V}$ is a *-vector space, then we call $v \in \mathcal{V}$ **self-adjoint** or **Hermitian**, provided that $v = v^*$. We let $\mathcal{V}_h$ denote the set of all self-adjoint elements of $\mathcal{V}$. It is easily seen that $\mathcal{V}_h$ is a real vector space.

Given an arbitrary element $v \in \mathcal{V}$, we set $Re(v) = \frac{v+v^*}{2}$ and $Im(v) = \frac{v-v^*}{2i}$, so that $Re(v)$ and $Im(v)$ are both self-adjoint elements and $v = Re(v) + iIm(v)$. This is often referred to as the **Cartesian decomposition of** v.

If, in addition, a *-vector space $\mathcal{V}$ is a complex algebra, then we require that an involution also satisfy,

$$(a \cdot b)^* = b^* \cdot a^*.$$

Thus, when we say that we have an involution on a complex algebra, we mean that all 4 properties are satisfied.

A **C*-algebra** is a Banach algebra that is equipped with an involution satsifying,

$$\|a^* \cdot a\| = \|a\|^2,$$

for every element. This equation is often called the **C*-property.**

We make no attempt to give a thorough course on C*-algebras in these notes, only a quick overview, emphasizing the things that we will need. For more complete treatments see [**?**], [**?**] and [**?**].

Proposition 2.1. *Let $\mathcal{A}$ be a C*-algebra, then for every $a \in \mathcal{A}$, $\|a\| = \|a^*\|$.*

Proof. We have that $\|a\|^2 = \|a^*a\| \le \|a^*\|\|a\|$, and canceling $\|a\|$ from each side yields, $\|a\| \le \|a^*\| <$ for every $a \in \mathcal{A}$. Applying this inequality to the element a^*, yields, $\|a^*\| \le \|a^{**}\| = \|a\|$, and equality follows. □

A few key examples of C*-algebras to keep in mind follow.

Example 2.2. *Let $\mathcal{H}$ denote a HIlbert space, $B(\mathcal{H})$ denote the algebra of bounded operators on $\mathcal{H}$, and for $T \in B(\mathcal{H})$, let T^* denote the usual adjoint operator. Then $B(\mathcal{H})$ is a C*-algebra.*

When $\mathcal{H} = \mathbb{C}^n$, then we will often identify $B(\mathcal{H})$ with the $n \times n$ complex matrices, M_n, in which case T^* is just the conjugate transpose of T.

Example 2.3. *Let X be a compact Hausdorff space and let $C(X)$ denote the algebra of continuous, complex-valued functions on X and for $f \in C(X)$, let f^* denote the function, $f^*(x) = \overline{f(x)}$. Then $C(X)$ equipped with the supremum norm, $\|f\| = \sup\{|f(x)| : x \in X\}$ is a C*-algebra.*

This last example can be generalized in two ways. If X is only locally compact and Hausdorff, then we can replace $C(X)$ with $C_0(X)$, the algebra of continuous, complex-valued functions that *vanish at infinity,* i.e., functions, f, such that for every $\epsilon > 0$, the set $\{x : |f(x)| \ge \epsilon\}$ is compact and obtain a C*-algebra. If X is completely regular(so in particular locally compact Hausdorff), then we may also replace $C(X)$ by $C_b(X)$, the algebra of continuous, complex-valued functions that are *bounded*, i.e., those functions, f, for which $\|f\| = \sup\{|f(x)| : x \in X\}$ is finite and $C_b(X)$ is a C*-algebra.

2.1. **Units and adjoining units.** We now take a look at properties of C*-algebras with units and how units may be adjoined.

Proposition 2.4. *Let $\mathcal{A}$ be a non-zero C*-algebra and assume that $e \in \mathcal{A}$ satisfies, $a \cdot e = e \cdot a = a$, for every $a \in \mathcal{A}$, i.e., that e is a two-sided unit, then $e = e^*$ and $\|e\| = 1$.*

Proof. For every $a \in \mathcal{A}$, we have that $(a \cdot e^*)^* = e^{**} \cdot a^* = e \cdot a^* = a^*$. Applying * to both sides yields, $a \cdot e^* = a$. Similarly, $e \cdot a = a$ and hence, e^* is also a two-sided unit. Hence, $e^* = e^* \cdot e = e$. Finally, $\|e\| = \|e^*e\| = \|e\|^2$, and hence, $\|e\|$ is either 0 or 1, and hence, $\|e\| = 1$. □

Proposition 2.5. *Let $\mathcal{A}$ be a C*-algebra, then for every $a \in \mathcal{A}$, we have that* $\|a\| = \sup\{\|ax\| : x \in \mathcal{A}, \|x\| \leq 1\} = \sup\{\|ya\| : y \in \mathcal{A}, \|y\| \leq 1\} = \sup\{\|yax\| : x, y \in \mathcal{A}, \|x\| \leq 1, \|y\| \leq 1\}$.

Proof. The equality is trivial for $a = 0$, so assume that $a \neq 0$. Since $\|ax\| \leq \|a\|\|x\| \leq \|a\|$, the supremum is smaller than the norm. Conversely, let $x_0 = a^*/\|a\|$, then $\|x_0\| = 1$, and hence, $\sup\{\|ax\| : \|x\| \leq 1\} \geq \|ax_0\| = \frac{\|aa^*\|}{\|a\|} = \|a\|$, and the first equality follows. The second equality follows by taking adjoints. The third equality follows by noting that

$$\sup\{\|yax\| : x, y \in \mathcal{A}, \|x\| \leq 1, \|y\| \leq 1\} = \sup\{\sup\{\|yax\| : x \in \mathcal{A}, \|x\| \leq 1\} : y \in \mathcal{A}, \|y\| \leq 1\} = \sup\{\|ax\| : x \in \mathcal{A}, \|x\| \leq 1\} = \|a\|.$$

□

Given a complex algebra, $\mathcal{A}$ if we let $\mathcal{A}_1 = \mathcal{A} \oplus \mathbb{C} = \{(a, \lambda) : a \in \mathcal{A}, \lambda \in \mathbb{C}\}$, then $\mathcal{A}_1$ is a complex vector space in the usual way and setting $(a_1, \lambda_1) \cdot (a_2, \lambda_2) = (a_1 a_2 + \lambda_2 a_1 + \lambda_1 a_2, \lambda_1 \lambda_2)$ can be easily shown to make $\mathcal{A}_1$ into a complex algebra with unit $e = (0, 1)$.

Moreover, the map $\iota : \mathcal{A} \to \mathcal{A}_1$, defined by $\iota(a) = (a, 0)$, is easily seen to be an algebra isomorphism onto its range. Identifying $a \in \mathcal{A}$ with $\iota(a)$ we see that $\mathcal{A}_1 = \{a + \lambda e : a \in \mathcal{A}, \lambda \in \mathbb{C}\}$.

The algebra $\mathcal{A}_1$ is called the **algebra obtained by adjoining a unit to $\mathcal{A}$.** Note that if $\mathcal{A}$ already had a unit, $u \in \mathcal{A}$, then $u \neq e \in \mathcal{A}_1$, but still $au = ua = a$, for every $a = \iota(a) \in \mathcal{A}_1$.

Also, note that if we set $(a, \lambda)^* = (a^*, \lambda^*)$, then this defines an involution on $\mathcal{A}_1$.

Proposition 2.6. *If $\mathcal{A}$ is a C*-algebra and we let $\mathcal{A}_1$ denote the algebra obtained from $\mathcal{A}$ by adjoining a unit, then $\mathcal{A}_1$ is a C*-algebra when we set* $\|(a, \lambda)\|_1 = \sup\{\|ax + \lambda x\| : x \in \mathcal{A}, \|x\| \leq 1\}$.

Proof. Note that for any $x \in \mathcal{A}, \|ax + \lambda x\| \leq \|(a, \lambda)\|_1 \|x\|$. Also, using Proposition 5, we have that $\|(a, \lambda)\|_1 = \sup\{\|yax + \lambda yx\| : x, y \in \mathcal{A}, \|x\| \leq 1, \|y\| \leq 1\} = \sup\{\|ya + \lambda y\| : y \in \mathcal{A}, \|y\| \leq 1\} = \sup\{\|a^* y^* + \bar{\lambda} y^*\| : y \in \mathcal{A}, \|y\| \leq 1\} = \|(a, \lambda)^*\|_1$.

We leave the details that this is a norm to the reader and only check the Banach algebra and C*-property. To this end note that,

$$\|(a, \lambda)(b, \mu)\|_1 = \sup\{\|a(bx + \mu x) + \lambda(bx + \mu x)\| : \|x\| \leq 1\} \leq \sup\{\|(a, \lambda)\|_1 \|bx + \mu x\| : \|x\| \leq 1\} = \|(a, \lambda)\|_1 \|(b, \mu)\|_1,$$

hence $\mathcal{A}_1$ is a Banach algebra.

To see the C*-property, note that $\|(a,\lambda)^*(a,\lambda)\|_1 \le \|(a^*,\bar\lambda)\|\|(a,\lambda)\|_1 = \|(a,\lambda)\|_1^2$. Also,

$$\|(a,\lambda)\|_1^2 = \sup\{\|(x^*a^* + x^*\bar\lambda)(ax+\lambda x)\| : \|x\| \le 1\} =$$
$$\sup\{\|(x^*a^*a + x^*a^*\lambda + x^*\bar\lambda a + x^*\bar\lambda\lambda)x\| : \|x\| \le 1\} \le$$
$$\sup\{\|x^*a^*a + x^*a^*\lambda + x^*\bar\lambda a + x^*\bar\lambda\lambda\| : \|x\| \le 1\} =$$
$$\sup\{\|x^*(a^*a + a^*\lambda + \bar\lambda a + \bar\lambda\lambda)\| : \|x\| \le 1\} =$$
$$\|(a^*a + a^*\lambda + a\bar\lambda, \bar\lambda\lambda)\|_1 = \|(a^*,\bar\lambda)(a,\lambda)\|_1$$

and the C*-property follows. □

Note that $\iota(ax+\lambda x) = (a,\lambda)(x,0)$.

2.2. **The Positive Cone of a C*-algebra.** If $\mathcal{A}$ is a unital C*-algebra, then an element $p \in \mathcal{A}$ is called **positive,** denoted $p \ge 0$ or $0 \le p$ provided that $p = p^*$ and the spectrum of $p, \sigma(p) \subseteq [0,+\infty)$. The set of all positive elements of $\mathcal{A}$ is denoted by $\mathcal{A}^+$. The following results give the key facts that we shall need about the positive elements.

Theorem 2.7. [**?**, Theorem VIII.3.6] *Let $\mathcal{A}$ be a unital C*-algebra and $a \in \mathcal{A}$, then the following are equivalent.*

(a) $a \ge 0$,
(b) $a = b^2$ *for some* $b \in \mathcal{A}_h$,
(c) $a = x^*x$ *for some* $x \in \mathcal{A}$,
(d) $a = a^*$ *and* $\|te - a\| \le t$ *for all* $t \ge \|a\|$,
(e) $a = a^*$ *and* $\|te - a\| \le t$ *for some* $t \ge \|a||$.

Proposition 2.8. [**?**, Proposition VIII.3.7] *Let $\mathcal{A}$ be a unital C*-algebra, then $\mathcal{A}^+$ is a closed cone.*

Proof. Recall that a **cone** is a convex set with the property that it is closed under scalar multiplication by non-negative scalars. Using (b), if $a \ge 0$, and $r \ge 0, r \in \mathbb{R}$ then $a = b^2$, and hence, $ra = (\sqrt{r}b)^2$, so that $ra \ge 0$.

To see that $\mathcal{A}^+$ is convex, it is enough to show that if $a, b \in \mathcal{A}^+$, then $(a+b)/2 \in \mathcal{A}^+$. We use (d), to see that for $t \ge \max\{\|a\|, \|b\|, \|a+b\|\}$, then $\|te - (a+b)/2\| \le \|(te-a)/2\| + \|(te-b)/2\| \le t/2 + t/2 = t$ and hence, by (e), $(a+b)/2 \ge 0$.

Finally, to see that $\mathcal{A}^+$ is closed, let $a_n \in \mathcal{A}^+$ be a sequence converging in norm to a. Since $a_n = a_n^*$, we have $a = a^*$. Also for $t \ge \sup\{\|a_n\|\}$, we have that $\|te - a\| = \lim_n \|te - a_n\| \le t$, and so by (e), $a \ge 0$. □

3. States and Pure States

Given a C*-algebra $\mathcal{A}$, a linear functional, $f : \mathcal{A} \to \mathbb{C}$ is called **positive,** provided that $f(p) \ge 0$, for every $p \in \mathcal{A}^+$. If $\mathcal{A}$ is a C*-algebra with unit e, then a **state** is a positive linear functional s on $\mathcal{A}$, such that, $s(e) = 1$.

Proposition 3.1. *Let $\mathcal{A}$ be a C^*-algebra with unit e and let $f : \mathcal{A} \to \mathbb{C}$ be a positive linear functional, then for every $x \in \mathcal{A}, f(x^*) = \overline{f(x)}$.*

Proof. First note that if $a = a^*$, then there exists $r \in \mathbb{R}$, such that $re + a \in \mathcal{A}^+$. Hence, $0 \leq f(re + a) = rf(e) + f(a)$, and it follows that $f(a) \in \mathbb{R}$. Now given any $x \in \mathcal{A}, f(x^*) = f(Re(x) - iIm(x)) = f(Re(x)) - if(Im(x)) = \overline{f(Re(x) + iIm(x))} = \overline{f(x)}$. □

Proposition 3.2 (Cauchy-Schwarz Inequality for states)**.** *Let $\mathcal{A}$ be a C^*-algebra with unit e, and let s be a state, then for any $x, y \in \mathcal{A}, |s(y^*x)|^2 \leq s(x^*x)s(y^*y)$ and s is a bounded linear functional and $\|s\| = 1$.*

Proof. First note that if $p \in \mathcal{A}^+$, then $\|p\|e - p \in \mathcal{A}^+$. Hence, $0 \leq s(\|p\|e - p) = \|p\| - s(p)$, so that $|s(p)| = s(p) \leq \|p\|$, for every $p \in \mathcal{A}^+$.

Now let $x, y \in \mathcal{A}$ and choose $\lambda \in \mathbb{C}, |\lambda| = 1$ so that $s(\lambda y^*x) = |s(y^*x)|$. Then for any $t \in \mathbb{R}$, we have that $0 \leq s((\lambda x - ty)^*(\lambda x - ty)) = s(x^*x) - 2t|s(y^*x)| + t^2 s(y^*y)$. So the roots of this polynomial are complex or repeated and hence(as in the proof of Cauchy-Schwartz), $|s(y^*x)|^2 \leq s(x^*x)s(y^*y)$.

Finally, for the last statement, taking $y = e$, we have that $|s(x)|^2 \leq s(x^*x) \leq \|x^*x\| = \|x\|^2$, and so $\|s\| \leq 1$, but $s(e) = 1$, implies that $\|s\| \geq 1$. □

There is also a converse to this last result.

Proposition 3.3. *Let $\mathcal{A}$ be a C^*-algebra with unit e and let $s : \mathcal{A} \to \mathbb{C}$ be a linear functional such that $\|s\| \leq 1$ and $s(e) = 1$, then s is a state.*

Proof. We must show that if $p \in \mathcal{A}^+$, then $s(p) \geq 0$. First we show that $s(p)$ is real. Write $s(p) = \alpha + i\beta$, with $\alpha, \beta \in \mathbb{R}$. Note that for $t \geq 0$, $\|p + it\beta e\|^2 = \|p\|^2 + t^2\beta^2$. Hence, $(t+1)^2\beta^2 \leq |s(p + it\beta e)|^2 \leq \|p + it\beta e\|^2 = \|p\|^2 + t^2\beta^2$. Canceling terms from both sides yields, $2t\beta^2 \leq \|p\|^2$ for all $t \geq 0$, which impies that $\beta = 0$.

Now we have that $|\alpha - \|p\|| = |s(p - \|p\|e)| \leq \|p - \|p\|e\| = \|p\|$, which implies that $\alpha \geq 0$. Thus, $s(p) \geq 0$. □

Given a unital C*-algebra $\mathcal{A}$ we let $S(\mathcal{A})$ denote the set of states on $\mathcal{A}$. By the above results this is a subset of the unit ball of the dual of $\mathcal{A}$ and hence, can be endowed with weak*-topology. When we refer to the **state space of** $\mathcal{A}$ we mean $S(\mathcal{A})$ endowed with this topology.

Proposition 3.4. *Let $\mathcal{A}$ be a unital C^*-algebra, then the state space of $\mathcal{A}$ is a weak*-closed, convex subset of the unit ball of the dual.*

Proof. Clearly convex combinations of states are states. Also, if a net of states converges in the weak*-topology to a linear functional, f, then $f(e) = 1$, and $\|f\| \leq 1$, so that f is a state. □

By the Krein-Milman theorem[**?**, Theorem V.7.4], $S(\mathcal{A})$ not only has extreme points but it is the closed, convex hull of its extreme points. We define the **pure states** on $\mathcal{A}$ to be the extreme points of the state space.

3.1. **States and the GNS Construction.** We outline/recall the famous Gelfand-Naimark-Segal construction.

Given a state s on a C*-algebra $\mathcal{A}$ with unit e, by the Cauchy-Schwarz inequality for states, setting $B(x,y) = s(y^*x)$ defines a positive, semidefinite sesquilinear form on $\mathcal{A}$.

If we let $\mathcal{N} = \{x \in \mathcal{A} : s(x^*x) = 0\}$, then it can be shown that $\mathcal{N}$ is a vector space and that for $a \in \mathcal{A}, a \cdot \mathcal{N} \subseteq \mathcal{N}$. From these facts one can see that there is a well-defined map $\hat{B}$ on the vector space $\mathcal{A}/\mathcal{N}$ defined by setting $\hat{B}(x+\mathcal{N}, y+\mathcal{N}) = B(x,y)$ and it is easy to check that $\hat{B}$ is a positive semidefinite sesquilinear form. Thus, $\mathcal{A}/\mathcal{N}$ is a *pre-Hilbert space* which after completion becomes a Hilbert space, which we denote by $\mathcal{H}_s$, to indicate its dependence on the state s.

There is a map $\pi_s : \mathcal{A} \to B(\mathcal{H}_s)$ defined on the dense subspace by $\pi_s(a)(x + \mathcal{N}) = ax + \mathcal{N}$. To see this note that we have that $\|\pi_s(a)(x + \mathcal{N})\|^2 = \|ax + \mathcal{N}\|^2 = s(x^*a^*ax)$. But $0 \leq x^*a^*ax \leq \|a^*a\|x^*x$, and hence, $0 \leq s(x^*a^*ax) \leq \|a^*a\|s(x^*x) = \|a\|^2\|x + \mathcal{N}\|^2$. Thus, $\pi_s(a)$ extends by continuity to a bounded operator on $\mathcal{H}_s$ of norm at most $\|a\|$.

It is easily seen that π_s is a unital homomorphism and that $\pi_s(a^*) = \pi_s(a)^*$, i.e., π_s is a ***-homomorphism.**

A unital *-homomorphism from a C*-algebra $\mathcal{A}$ into $B(\mathcal{H})$ for some Hilbert space $\mathcal{H}$ is often called a **representation of $\mathcal{A}$ on $\mathcal{H}$.** The map $\pi_s : \mathcal{A} \to B(\mathcal{H}_s)$ is called the **Gelfand-Naimark-Segal(or GNS) representation corresponding to the state s.**

Note that if we set $\eta = e + \mathcal{N}$, then $\|\eta\| = s(e) = 1$ and we recover the state from the representation by the formula,

$$s(a) = \langle \pi_s(a)\eta, \eta \rangle.$$

Conversely, if we start with any representation $\pi : \mathcal{A} \to B(\mathcal{H})$, and any unit vector $v \in \mathcal{H}$, then $s(a) = \langle \pi(a)v, v \rangle$ is a state.

The GNS representations have one additional property. A representation $\pi : \mathcal{A} \to B(\mathcal{H})$ is called **cyclic** if there exists a vector $h \in \mathcal{H}$, such that the set $\pi(\mathcal{A})h \doteq \{\pi(a)h : a \in \mathcal{A}\}$ is dense in $\mathcal{H}$, and in this case h is called a **cyclic vector.** If we let $\eta = e + \mathcal{N}$, then the set $\pi_s(\mathcal{A})\eta = \{\pi_s(a)\eta : a \in \mathcal{A}\} = \{a+\mathcal{N} : a \in \mathcal{A}\}$ is a dense subset of $\mathcal{H}_s$. Thus, the GNS representations are always cyclic.

In fact cyclicity characterizes the GNS representation.

Proposition 3.5. *Let $\mathcal{A}$ be a C^*-algebra with unit e, and let $\pi : \mathcal{A} \to B(\mathcal{H})$ be a representation with a unit cyclic vector v. If we let $s(a) = \langle \pi(a)v, v \rangle$ denote the corresponding state, then there exists a unitary, $U : \mathcal{H}_s \to \mathcal{H}$, with $U\pi_s(a)\eta = \pi(a)v$.*

Proof. One has that $\|\pi_s(a)\eta\|^2 = \|a + \mathcal{N}\|^2 = s(a^*a) = \langle \pi(a^*a)v, v \rangle = \|\pi(a)v\|^2$. This equation shows that the map U is well-defined and an isometry. The rest follows readily. □

The GNS representations are used to prove that for every C*-algebra $\mathcal{A}$, there exists a Hilbert space $\mathcal{H}$ and an isometric representation of $\mathcal{A}$ on $\mathcal{H}$. We already have all but one of the results necessary to prove this facts.

Lemma 3.6. *Let $\mathcal{A}$ be a unital C*-algebra and let $p \in \mathcal{A}^+$, then there exists a state s such that $\|p\| = s(p)$.*

Proof. Let $\mathcal{S}$ denote the span of e and p and define a linear functional, $f; \mathcal{S} \to \mathbb{C}$ by $f(\alpha e + \beta p) = \alpha + \beta\|p\|$. We use the fact that $\|\alpha e + \beta p\| = \sup\{|\alpha + \beta\lambda| : \lambda \in \sigma(p)\}$, where $\sigma(p)$ denotes the spectrum of p and that $\|p\| \in \sigma(p)$.

¿From this is follows that $|f(\alpha e + \beta p)| = |\alpha + \beta\|p\|| \leq \|\alpha e + \beta p\|$, and so f is a contraction. By the Hahn-Banach theorem, we may extend f to a contractive linear functional, $s : \mathcal{A} \to \mathbb{C}$, but since $s(e) = 1$ and $\|s\| = 1$, s is a state. □

Theorem 3.7 (Gelfand-Naimark-Segal Representation)**.** *Let $\mathcal{A}$ be a C*-algebra, then there exists a Hilbert sapce $\mathcal{H}$ and an isometric *-homomorphism $\pi : \mathcal{A} \to B(\mathcal{H})$. Moreover, if $\mathcal{A}$ is unital, then one may chose π to be unital too.*

Proof. We only do the case that $\mathcal{A}$ is a unital C*-algebra, the non-unital result follows by adjoining a unit to $\mathcal{A}$.

For each state s on $\mathcal{A}$, let π_s and $\mathcal{H}_s$ denote the GNS representation and corresponding Hilbert space. Let $\mathcal{H} = \sum_s \oplus \mathcal{H}_s$, be the orthogonal direct sum over all states and let $\pi = \sum_s \oplus \pi_s : \mathcal{A} \to B(\mathcal{H})$ be the representation that is the direct sum of al the GNS representations. Then π is a unital *-homomorphism and for each $a \in \mathcal{A}$, we have that $\|a\|^2 \geq \|\pi(a)\|^2 = \|\pi(a^*a)\| = \sup_s\{\|\pi_s(a^*a)\|\} \geq \|a^*a\|$, where the last inequality follows from the lemma. Hence, π is an isometry. □

The representation $\pi = \sum_s \pi_s$ is called the **universal representation of $\mathcal{A}$**.

We now look at what distinguishes the pure states. Given a representation $\pi : \mathcal{A} \to B(\mathcal{H})$ of a C*-algebra, a subspace $\mathcal{H}_1 \subseteq \mathcal{H}$ is called **invariant for** $\pi(\mathcal{A})$ provided that $\pi(\mathcal{A})\mathcal{H}_1 \subseteq \mathcal{H}_1$ and **reducing for** $\pi(\mathcal{A})$ provided that the orthogonal projection onto $\mathcal{H}_1$, say P_1, commutes with $\pi(\mathcal{A})$, i.e., $P_1\pi(a) = \pi(a)P_1$, for every $a \in \mathcal{A}$. A representation is called **irreducible** if the only reducing subspaces are $\mathcal{H}$ and (0).

Proposition 3.8. *Let $\pi : \mathcal{A} \to B(\mathcal{H})$ be a representation of the C*-algebra $\mathcal{A}$ and let $\mathcal{H}_1$ be a subspace. Then $\mathcal{H}_1$ is invariant if and only if $\mathcal{H}_1$ is reducing.*

Proof. Clearly, if $\mathcal{H}_1$ is reducing, then it is invariant. Conversely, if $\mathcal{H}_1$ is an invariant subspace, then for every $a \in \mathcal{A}$, we have that $P_1\pi(a)P_1 = \pi(a)P_1$. But then it follows that $P_1\pi(a) = [\pi(a)^*P_1]^* = [\pi(a^*)P_1]^* = [P_1\pi(a^*)P_1]^* = P_1\pi(a^*)^*P_1 = P_1\pi(a)P_1 = \pi(a)P_1$, and so P_1 commutes. Hence, $\mathcal{H}_1$ is reducing. □

Proposition 3.9. *Let $\mathcal{A}$ be a unital C*-algebra and let s be a state. Then s is a pure state if and only if π_s is an irreducible representation.*

Proof. We prove only one implication, namely, that if s is pure, then π_s is irreducible. For the proof of the converse see [**?**, Theorem I.9.8].

To this end assume that π_s is reducible and let $\mathcal{H}_1$ be a non-trivial reducing subspace. Let P_1 denote the projection onto $\mathcal{H}_1$ and let $P_2 = P_1^{\perp}$ and let η denote the cyclic vector for π_s. If $\eta = P_1\eta$, then it easily follows that $\mathcal{H}_1 = \mathcal{H}_s$. Hence, $\eta \neq P_1\eta$. Similarly, $\eta \neq P_2\eta$.

Thus, $P_1\eta \neq 0$, and $P_2\eta \neq 0$. Let $\eta_i = \frac{P_i\eta}{\|P_i\eta\|}, i = 1,2$ and define states by $s_i(a) = \langle \pi_s(a)\eta_i, \eta_i\rangle, i = 1,2$. It is easily checked that $s(a) = \|\eta_1\|^2 s_1(a) + \|\eta_2\|^2 s_2(a)$, a convex combination.

Now since s is pure, we must have that $s = s_1 = s_2$. Thus, in particular there exists a unitary, $U : \mathcal{H}_s \to \mathcal{H}_1$, where $\mathcal{H}_1$ is the closed cyclic subspace generated by $\pi_s(\mathcal{A})\eta_1$, satisfying, $U\pi_s(a)\eta = \pi_s(a)\eta_1 = \|P_1\eta\|^{-1}\pi_s(a)P_1\eta = \|P_1\eta\|^{-1}P_1\pi_s(a)\eta$. Since the vectors $\{\pi_s(a)\eta : a \in \mathcal{A}\}$ span the Hilbert space, we have that $U = \|P_1\eta\|^{-1}P_1$, but the left side is an isometry while the right side is not.

This contradiction completes the proof. □

3.2. **States and Pure States on C(X).** Let X denote a compact, Hausdorff space. By the Riesz Representation Theorem, every bounded linear functional on $C(X)$ is given by integration against a regular, complex-valued Borel measure on X and the positive linear functionals are integration against a positive regular, Borel measure. Thus, a state s corresponds to integration against a positive, regular Borel measures μ with $1 = s(e) = \int_X 1 d\mu$ and hence, $\mu(X) = 1$.

To get $\mathcal{H}_s$ we complete $C(X)$ in the inner product, $\langle f, g\rangle = s(\bar{g}f) = \int_X \bar{g}f d\mu$. Thus, $\mathcal{H}_s = L^2(X,\mu)$! Moreover, $\pi_s(f) = M_f$ the operator of multiplication by f on $L^2(X,\mu)$.

Now given any measurable set E, if we let P_E denote the operator given by multiplication the characteristic function of E, then P_E is an orthogonal projection that commutes with $\pi_s(C(X))$. Thus, P_E defines a reducing subspace.

Hence, s is pure if and only if for every E either $P_E = I$ or $P_E = 0$. This last condition holds if and only if the support of the measure μ consists of a single point, that is, if and only if there exists a point $x \in X$, such that $\mu = \delta_x$ where δ_x is the measure defined by

$$\delta_x(E) = \begin{cases} 1 & x \in E \\ 0 & x \notin E \end{cases}.$$

Note that in this case the state is given by $s(f) = \int_X f d\delta_x = f(x)$, the functional of evaluation at x.

Thus we see that the pure states on $C(X)$ are all *-homomorphisms given by evaluation at points.

3.3. **States and Pure States on M_n.** We now look at the states and pure states on $M_n = B(\mathbb{C}^n)$. First note that every vector in $\mathbb{C}^n$ is cyclic for M_n, i.e., the identity representation of M_n is irreducible. Thus, if we take any unit vector $v \in \mathbb{C}^n$, then the state $s_v(A) = \langle Av, v\rangle$ is pure. These states are called **vector states.**

On the other hand, it can be shown that if $\pi : M_n \to B(\mathcal{H})$ is any representation, then up to conjugation by a unitary, $\mathcal{H}$ is a direct sum of copies of $\mathbb{C}^n$(so that if the dimension of $\mathcal{H}$ is finite, then it is divisible by n) and $\pi(A)$ is a direct sum of the same number of copies of A(so that $\pi(A) = A \oplus A \oplus \ldots$). To see this claim, let $E_{i,j}$ denote the canonical matrix units for M_n, let $\mathcal{K} = \pi(E_{1,1})\mathcal{H}$ and choose an orthonormal basis $\{f_\alpha\}_{\alpha \in A}$ for $\mathcal{K}$ where A is some index set. If we let $e_{i,\alpha} = \pi(E_{i,1}f_\alpha$, and set $\mathcal{H}_\alpha = span\{e_{i,\alpha} : 1 \le i \le n\}$, then each of these spaces is n-dimensional and reducing for $\pi(M_n)$. The restrictions of π to each of these spaces is easily seen to be the identity representation of M_n with respect to the orthonormal basis, $\{e_{i,\alpha} : 1 \le i \le n\}$.

These representations are reducible, unless $dim(\mathcal{K}) = 1$, in which case we are back to the first example. Thus, every pure state on M_n is a vector state.

This decomposition of representations does yield a representation of arbitrary states. For if we are given a state s and π_s, then $\pi_s(A) = A \oplus A \oplus \ldots$ and relative to this decomposition $\eta = \eta_1 \oplus \eta_2 \oplus \ldots$, with $1 = \|\eta\|^2 = \|\eta_1\|^2 + \|\eta_2\|^2 + \ldots$ Thus, if we let $v_i = \eta_/\|\eta_i\|$, then

$$s(A) = \langle A\eta_1, \eta_1\rangle + \langle A\eta_2, \eta_2\rangle + \ldots = \|\eta_1\|^2 s_{v_1}(A) + \|\eta_2\|^2 s_{v_2}(A) + \ldots$$

and we have expressed s as a convex series of pure states.

3.4. **States and Pure States on $B(\mathcal{H})$.** The same argument as for M_n, shows that if we take any unit vector, $v \in \mathcal{H}$, then setting $s_v(A) = \langle Av, v\rangle$ defines a pure state. However, when $\mathcal{H}$ is infinite dimensional, then these are not all of the pure states! In fact, in some sense, it is the fact that there are other pure states on $B(\mathcal{H})$ that we shall see is at the heart of the Kadison-Singer problem.

To see that there exist other pure states, we need only recall that the compact operators, $\mathcal{K}(\mathcal{H})$ form a two-sided ideal in $B(\mathcal{H})$ and so we may form a quotient algebra, $\mathcal{Q}(\mathcal{H}) = B(\mathcal{H})/\mathcal{K}(\mathcal{H})$. This quotient algebra is called the **Calkin algebra.**

The Calkin algebra is itself a C*-algebra(quotients of C*-algebras by two-sided ideals are always C*-algebras) and so there exist states and pure states on $\mathcal{Q}(\mathcal{H})$. Note that every state $\hat{s}$ on $\mathcal{Q}(\mathcal{K})$ by composition with the quotient map yields a state $s(A) = \hat{s}(A + \mathcal{K}(\mathcal{H}))$ on $B(\mathcal{H})$ that is 0 on $\mathcal{K}(\mathcal{H})$. Moreover, it is easily seen that if $\hat{s}$ is a pure state, then s is also pure.

Since the vector states obtained above do not vanish on $\mathcal{K}(\mathcal{H})$, the states on $\mathcal{Q}(\mathcal{H})$ yield new non-vector states and non-vector pure states.

Thus, we see that there are two types of pure states on $B(\mathcal{H})$, vector states and states that are the composition of a pure state on the Calkin algebra with the quotient map.

There is still a great deal that is not understood about pure states on the Calkin algebra and the corresponding irreducible representations. Remarkably, some results about the representations of $\mathcal{Q}(\mathcal{H})$ and hence of $B(\mathcal{H})$ depend upon the axioms of set theory. Some of the deepest work can be found in [**?**], [**?**], [**?**], and [**?**].

3.5. **The Generalized Kadison-Singer Problem.** We now return to the Kadison-Singer problem and see what the above knowledge of states and pure states tells us.

Proposition 3.10. *Let $\mathcal{B}$ be a unital C^*-algebra and let $\mathcal{A} \subseteq \mathcal{B}$ be a subalgebra that contains the unit of $\mathcal{B}$. Then every state on $\mathcal{A}$ extends to a state on $\mathcal{B}$.*

Proof. Let $s : \mathcal{A} \to \mathbb{C}$ be a state, then $\|s\| = 1$, and so by the Hahn-Banach theorem there exists a linear map $\tilde{s} : \mathcal{B} \to \mathbb{C}$ extending s with $\|\tilde{s}\| = 1$. But since $\tilde{s}(e) = s(e) = 1, \tilde{s}$ is also a state. □

The following result is implicitly contained in [**?**].

Proposition 3.11. *Let $\mathcal{B}$ be a unital C^*-algebra and let $\mathcal{A} \subseteq \mathcal{B}$ be a subalgebra that contains the unit of $\mathcal{B}$. Then the following are equivalent:*

- *every pure state on $\mathcal{A}$ extends to a unique pure state on $\mathcal{B}$,*
- *every pure state on $\mathcal{A}$ extends to a unique state on $\mathcal{B}$.*

Proof. Fix a pure state $s : \mathcal{A} \to \mathbb{C}$ and let $\mathcal{C}$ denote the set of all states on $\mathcal{B}$ that extend s. Then it is easy to see that $\mathcal{C}$ is a weak*-closed, convex subset of the unit ball of the dual space of $\mathcal{B}$.

Let $\tilde{s} \in \mathcal{C}$ be any state on $\mathcal{B}$ that extends s. If we express $\tilde{s}$ as a convex combination of states on $\mathcal{B}$, then the restriction of each of these states to $\mathcal{A}$ expresses s as a convex combination of states. But since s is an extreme point, each of these restricted states must be s. Thus, whenever we express $\tilde{s} \in \mathcal{C}$ as a convex combination of states on $\mathcal{B}$, each of those states is necessarily also in $\mathcal{C}$.

Hence, every extreme point of $\mathcal{C}$ is a pure state.

Now assume the first statement. If there exists more than one extension of s, then there is more than one point in $\mathcal{C}$ and hence, by Krein-Milman $\mathcal{C}$ would have at least two extreme points. But both of these extreme points would be pure states that extend s.

If we assume the second statement, then $\mathcal{C}$ is a singleton and that singleton is an extreme point of $\mathcal{C}$ and so necessarily a pure state. □

This result leads to the following equivalent re-formulation of the Kadison-Singer problem:

The Kadison-Singer Problem: Let $\mathcal{H}$ be a separable Hilbert space and let $\mathcal{D} \subseteq B(\mathcal{H})$ be a discrete MASA. Does every pure state on $\mathcal{D}$ extend to a unique state on $B(\mathcal{H})$?

It also leads to the following generalizations of the Kadison-Singer problem.

Let $\mathcal{B}$ be a unital C*-algebra and let $\mathcal{A} \subseteq \mathcal{B}$ be a subalgebra that contains the unit of $\mathcal{B}$. Find necessary and sufficient conditions on the pair of C*-algebras so that every pure state on $\mathcal{A}$ extends to a unique state on $\mathcal{B}$.

Since states are the same as linear functionals of norm one which send the identity to 1, this question makes sense even when $\mathcal{B}$ is not a C*-algebra. Consequently, the following problem is also studied.

Let $\mathcal{A}$ be a unital C*-algebra, $\mathcal{X}$ a Banach space with $\mathcal{A} \subseteq \mathcal{X}$. Find necessary and sufficient conditions on the pair so that every pure state on $\mathcal{A}$ extends uniquely to a contractive linear functional on $\mathcal{X}$.

Since MASA's are abelian C*-algebras(as we shall see in the next section), these last two questions are most often studied in the case where $\mathcal{A} = C(X)$ for some compact, Hausdorff space X, in which case the pure state is just evaluation at a point of X.

4. Discrete and continuous MASA's

A subalgebra $\mathcal{A} \subseteq B(\mathcal{H})$ is **a maximal abelian self-adjoint subalgebra or MASA** provided that if $\mathcal{A} \subseteq \mathcal{B} \subseteq B(\mathcal{H})$ with $\mathcal{B}$ an abelian self-adjoint subalgebra, then $\mathcal{A} = \mathcal{B}$. These always exist by Zorn's Lemma.

Definition 4.1. *Let $\mathcal{S} \in B(\mathcal{H})$ be a non-empty set. Then* **commutant** *of $\mathcal{S}$ is*

$$\mathcal{S}' = \{T \in B(\mathcal{H}) : TS = ST, \text{ for all } S \in \mathcal{S}.\}$$

The following are some basic properties of commutants that we will make use of.

1) If $\mathcal{S} \subseteq \mathcal{T}$, then $\mathcal{T}' \subseteq \mathcal{S}'$,
2) $\mathcal{S}'$ is a unital subalgebra of $B(\mathcal{H})$.
3) $\mathcal{S}'$ is norm-closed.
4) We call a set $\mathcal{S}$ self-adjoint if whenever $S \in \mathcal{S}$, then $S^* \in \mathcal{S}$. If $\mathcal{S}$ is self-adjoint, then $\mathcal{S}'$ is self-adjoint. to see this note that $T \in \mathcal{S}'$ implies $TS = ST$ for all $S \in \mathcal{S}$ and so $S^*T^* = T^*S^*$ for all $S^* \in \mathcal{S}$ which shows that $T^* \in \mathcal{S}'$.
5) If $\mathcal{S}$ is abelian, then $\mathcal{S} \subseteq \mathcal{S}'$.

Proposition 4.2. *Let $\mathcal{A} \subseteq B(\mathcal{H})$ be an abelian self-adjoint algebra. Then $\mathcal{A}$ is a MASA if and only if $\mathcal{A} = \mathcal{A}'$.*

Proof. If $B \in \mathcal{A}'$, then $BA^* = A^*B$ for all $A \in \mathcal{A}$. This implies $AB^* = B^*A$ for all $B^* \in \mathcal{A}'$. This implies $Re(B) = \dfrac{B + B^*}{2} \in \mathcal{A}'$. Let $\mathcal{B}$ be the algebra generated by $\{\mathcal{A}, Re(B)\}$. $\mathcal{B}$ is an abelian, self-adjoint algebra in $B(\mathcal{H})$, and $\mathcal{A} \subseteq \mathcal{B}$. By the maximality of $\mathcal{A}$ we get $\mathcal{A} = \mathcal{B}$. We have shown $Re(B) \in \mathcal{A}$ and a similar argument whows $Im(B) \in \mathcal{A}$. Therefore, $\mathcal{A}' \subseteq \mathcal{A}$, but $\mathcal{A} \subseteq \mathcal{A}'$ by our earlier remark.

If $\mathcal{A} \subseteq \mathcal{B} \subseteq B(\mathcal{H})$, then $\mathcal{B}' \subseteq \mathcal{A}' = \mathcal{A}$. Since $\mathcal{B}$ is abelian, $\mathcal{B} \subseteq \mathcal{B}' \subseteq \mathcal{A}$. Therefore, $\mathcal{B} = \mathcal{A}$, which implies $\mathcal{A}$ is maximal, i.e. $\mathcal{A}$ is a MASA. □

Corollary 4.3. *If $\mathcal{A} \subseteq B(\mathcal{H})$ is a MASA, then $\mathcal{A} = \mathcal{A}' = \mathcal{A}''$.*

Proof. $\mathcal{A} = \mathcal{A}' \Rightarrow (\mathcal{A}')' = \mathcal{A}' = \mathcal{A}$. □

Example 1 Consider the Hilbert space $\mathcal{H} = \ell^2(\mathbb{N})$ with its standard orthonormal basis $\{e_j : j \in \mathbb{N}\}$. If $T \in B(\ell^2(\mathbb{N}))$, then we may identify T with a matrix $T = (t_{i,j})_{i,j\in\mathbb{N}}$, where $t_{i,j} = \langle Te_j, e_i\rangle$. An operator $T \in B(l^2(\mathbb{N}))$ is called **diagonal** if $t_{i,j} = 0$, for all $i \neq j$. Let $\mathcal{D}$ denote the set of diagonal operators on $\ell^2(\mathbb{N})$. The algebra of diagonal operators $\mathcal{D}$ is **self-adjoint**. Let $D \in \mathcal{D}$, then $\langle De_j, e_i\rangle = 0$ for all $i \neq j$. If follows that $\langle e_j, D^*e_i\rangle = 0$ for all $i \neq j$ and so $\langle D^*e_i, e_j\rangle = 0$ for all $i \neq j$ and so $D^* \in \mathcal{D}$. To see that $\mathcal{D}$ is **abelian** we note that if $T = (t_{i,j})$ and $R = (r_{i,j})$ in $B(l^2(\mathbb{N}))$, then $TR = (\sum_{k=1}^{\infty} t_{ik}r_{kj})_{i,j=\mathbb{N}}$. Firstly, each row and column of the matrix of such a bounded operator is in $\ell^2(\mathbb{N})$. We have, $Te_j = \sum_{i=1}^{\infty}\langle Te_j, e_i\rangle e_i$. The Parseval identity yields

$$\sum_{i=1}^{\infty} |\langle Te_j, e_i\rangle|^2 < \infty. \tag{1}$$

Similarly, we have $T^*e_j = \sum_{i=1}^{\infty}\langle T^*e_j, e_i\rangle e_i$ and so

$$\sum_{i=1}^{\infty} |\langle T^*e_j, e_i\rangle|^2 < \infty. \tag{2}$$

By (**??**), $\sum_{i=1}^{\infty} |\langle Te_j, e_i\rangle|^2 = \sum_{i=1}^{\infty} |t_{ij}|^2$, and by (**??**), $\sum_{i=1}^{\infty} |\langle T^*e_j, e_i\rangle|^2 = \sum_{i=1}^{\infty} |t_{ji}|^2$. This shows for any $T, R \in B(l^2(\mathbb{N}))$ that $\sum_{k=1}^{\infty} t_{ik}r_{kj}$ converges. We compute

$$\begin{aligned}\langle TRe_j, e_i\rangle = \langle T(\sum_{k=1}^{\infty}\langle Re_j, e_k\rangle e_k), e_i\rangle &= \langle T(\sum_{k=1}^{\infty} r_{kj}e_k, e_i\rangle \\ &= \langle\sum_{k=1}^{\infty} r_{kj}Te_k, e_i\rangle = \sum_{k=1}^{\infty}\langle Te_k, e_i\rangle r_{kj} = \sum_{k=1}^{\infty} t_{ik}r_{kj}.\end{aligned}$$

Finally to show that $\mathcal{D}$ is a **maximal** we need only to show $\mathcal{D} = \mathcal{D}'$. Since $\mathcal{D}$ is abelian, $\mathcal{D} \subseteq \mathcal{D}'$. If $T \in \mathcal{D}'$, then $TD = DT$ for all $D \in \mathcal{D}$. In particulat T must commute with $E_{i,i}$ for all $i \in \mathbb{N}$. We have,

$$TE_{i,i} = \begin{pmatrix} 0 & \cdots & 0 & t_{1,i} & 0 & \cdots \\ 0 & \cdots & 0 & t_{2,i} & 0 & \cdots \\ \vdots & \cdots & \vdots & \vdots & \vdots & \cdots \end{pmatrix}$$

and

$$E_{i,i}T = \begin{pmatrix} 0 & \cdots & 0 & \cdots \\ \vdots & \cdots & \vdots & \cdots \\ t_{i,1} & \cdots & t_{i,i} & \cdots \\ 0 & \cdots & 0 & \cdots \\ \vdots & \cdots & \vdots & \cdots \end{pmatrix}.$$

On comparing these two matrices we see that $t_{i,j} = 0$ for $i \neq j$. Therefore $T \in \mathcal{D} \Rightarrow \mathcal{D}' \subseteq \mathcal{D}$, hence $\mathcal{D}$ is a MASA.

Note: For any $D = \text{diag}(d_{i,i}) \in \mathcal{D}$, we have $|d_{ii}| = |\langle De_i, e_i\rangle| \leq \|D\|$. Hence, $\sup_i\{|d_{ii}|\} \leq \|D\|$. On the other hand $D(\sum_i \alpha_i e_i) = \sum_i \alpha_i d_{ii} e_i$, which implies $\|D(\sum_i \alpha_i e_i)\|^2 = \sum_i |\alpha_i|^2 |d_{ii}|^2 \leq \sup_i\{|d_{ii}|\} \sum_i |\alpha_i|^2 = \sup_i\{|d_{ii}|\} \| \sum_i \alpha_i e_i\|^2$. Therefore, $\|D\| \leq \sup_i\{|d_{ii}|\}$ and $\|D\| = \sup\{|d_{ii}| : i \in \mathbb{N}\}$.

This allows us to identify $\ell^\infty(\mathbb{N}) = C_b(\mathbb{N})$, which is an abelian C* algebra with $\mathcal{D}$. Consider the representation $\pi : \ell^\infty(\mathbb{N}) \to B(\ell^2(\mathbb{N}))$ given by $\pi((\alpha_j)) = D$, where $D = \text{diag}(d_{jj})$ with $d_{jj} = \alpha_j$. We will often identify $\mathcal{D}$ with $\ell^\infty(\mathbb{N})$.

Example 2 Consider the Hilbert space $\mathcal{H} = L^2([0,1], \lambda)$, where λ is Lebesgue measure. Given $f \in L^\infty([0,1])$ define the multiplication operator on $L^2([0,1])$ by $M_f(g) = f \cdot g$. Denote by $\mathcal{M} = \{M_f | f \in L^\infty[0,1]\} \subseteq B(L^2[0,1])$. Recall that $\|M_f\| = \text{esssup}\|f| = \|f\|_\infty$. It can be checked easily that $\mathcal{M}$ is an **abelian, self-adjoint subalgebra** (recall $(M_f)^* = M_{\overline{f}}$).

We claim that $\mathcal{M}$ is maximal, i.e. a MASA. Let $T \in \mathcal{M}'$. Set $g = T(1)$, where 1 is the function constantly equal to 1. For $f \in L^\infty \subseteq L^2, T(f) = T(M_f(1)) = M_f T(1) = M_f(g) = f \cdot g$. Therefore, $T(f) = f \cdot g$ for all $f \in L^\infty$. It remains to be shown that $g \in L^\infty$. Let $E_n = \{x : |g(x)| \geq n\}$, let $f = \chi_{E_n} \in L^\infty$. Then $\|T(f)\|^2 = \|g \cdot f\|^2 = \int_{[0,1]} |g\chi_{E_n}|^2 d\lambda \geq \int n^2 |\chi_{E_n}|^2 d\lambda = n^2 \int |\chi_{E_n}|^2 d\lambda = n^2 \|f\|^2 \Rightarrow \|T\|^2 \geq n^2$ provided $\|f\|^2 \neq 0$. Therefore $m(\{x : |g(x) \geq n\}) = 0$ whenever $n > \|T\|$. Hence, $g \in L^\infty$. Note $T(f) = M_g(f)$ for all $f \in L^\infty$. Since both T and M_f are bounded operators and L^∞ is dense in L^2, we get $T = M_g$. Hence, $\mathcal{M}' \subseteq \mathcal{M} \subseteq \mathcal{M}'$.

Definition 4.4. *Let $\mathcal{A} \subseteq B(\mathcal{H})$ be a unital C^* subalgebra, a projection $P = P^2 = P^* \in \mathcal{A}$ is called **minimal**, if 0 $\neq P$ and whenever $E = E^2 = E^* \in \mathcal{A}$ with $0 \leq E \leq P$, then either $E = 0$ or $E = P$.*

Definition 4.5. *A **MASA** $\mathcal{A} \subseteq B(\mathcal{H})$ is called* **discrete** *or* **atomic** *if it is the commutant of the set of its minimal projections.*

Definition 4.6. *A **MASA** $\mathcal{A} \subseteq B(\mathcal{H})$ is called* **continuous** *if it has no minimal projections.*

Example 3 Let $\mathcal{D} \subseteq \ell^\infty(\mathbb{N}) \subseteq B(\ell^2(\mathbb{N}))$, each $E_{ii} \in \mathcal{D}$ is a minimal projection and $\{E_{ii} : i \in \mathbb{N}\}' = \mathcal{D}$. Thus, $\mathcal{D}$ is a discrete MASA.

Example 4 Let $\mathcal{M} = L^\infty[0,1] \subseteq B(L^2[0,1])$. We know that $P \in \mathcal{M}$, $P^2 = P = P^*$ iff $P = M_{\chi_E}$, for some Borel measurable subset $E \subseteq [0,1]$. If

$P \neq 0$, then $m(E) \neq 0$. If $m(E) \neq 0$, then there exists Borel measurable sets E_1 and E_2 such that $E = E_1 \cup E_2$ and $m(E_i) \neq 0$.

Since $0 < m(E_1) < m(E)$, we have $0 \leq M_{\chi_{E_1}} \leq M_{\chi_E}$ which gives a non-trivial projection $M_{\chi_{E_1}}$. Therefore, $\mathcal{M} = L^\infty[0,1]$ is a continuous MASA.

Proposition 4.7 (Da, II.1.2)**.** *Let $\mathcal{A}$ be a unital C^* algebra, $\pi : \mathcal{A} \to B(\mathcal{H})$ a unital $*$-homomorphism. Assume that $\mathcal{H}$ is seperable. There exists an at most countable collection of orthogonal subspaces $\{\mathcal{H}_n\}$ of $\mathcal{H}$ with $\mathcal{H} = \sum_n \oplus \mathcal{H}_n$ such that each $\mathcal{H}_n$ is reducing for π, and there exists $h_n \in \mathcal{H}_n$ such that $\overline{[\pi(a)h_n]} = \mathcal{H}_n$.*

Proof. Given any $x \in \mathcal{H}$, x$\neq$ 0. Look at $\mathcal{H}_x = [\pi(a)x]^-$, then $\mathcal{H}_x$ is invariant for π and hence reducing.

On ${\mathcal{H}_x}^\perp$, if we pick any $y \in {\mathcal{H}_x}^\perp, [\pi(a)y]^- = \mathcal{H}_y$ will be othogonal to $\mathcal{H}_x$.

By Zorn's lemma we may pick maximal set $\{X_\alpha\}_{\alpha \in A}$ such that $\mathcal{H}_{x_\alpha} = [\pi(a)x_\alpha]^-$ are all orthogonal for $x_\alpha \neq x_\beta$.
Since $x_\alpha \in \mathcal{H}_{x_\alpha}$, $x_\beta \in \mathcal{H}_{x_\beta}$ implies that $x_\alpha \perp x_\beta$. and so A is atmost countable.

Lastly, $\sum_{\alpha \in A} \bigoplus \mathcal{H}_{x_\alpha} = \mathcal{H}$, by maximality. □

Lemma 4.8. *Let $\mathcal{A} \subseteq B(\mathcal{H})$ MASA, where $\mathcal{H}$ is seperable. Then there exists $x \in \mathcal{H}$ such that $[\mathcal{A}x]^- = \mathcal{H}_x$.*

Proof. Write $\mathcal{H} = \sum_n \oplus \mathcal{H}_{x_n}$ where each $\mathcal{H}_{x_n} = [\mathcal{A}x_n]^-, \ \|x_n\| = 1$.

Let P_n= projection on $\mathcal{H}_{x_n} = P_{H_{x_n}}$ then $P_n \in \mathcal{A}' = \mathcal{A}$.

Let $x = \sum_{n=1}^\infty \frac{1}{2^n} x_n, \ [\mathcal{A}x]^- \supseteq [\mathcal{A}P_n x]^- = [\mathcal{A}x_n]^- = \mathcal{H}_{x_n}$ which implies that $[\mathcal{A}x]^- \supseteq \sum \oplus \mathcal{H}_n = \mathcal{H}$. □

Theorem 4.9 (MASA Representation theorem)**.** *Let $\mathcal{H}$ be a seperable Hilbert space, $\mathcal{A} \subseteq B(\mathcal{H})$ MASA. Then there exists a compact Hausdroff space X, a Borel measure μ on X with $\mu(X) = 1$ and a unitary $U : L^2(X, \mu) \to \mathcal{H}$ such that $\mathcal{A} = \{UM_fU^* \ : \ f \in L^\infty(X, \mu)\}$.*

We sketch the proof of this result.

Proof. Let $\mathcal{A} \subseteq B(\mathcal{H})$ MASA, $\mathcal{H}$ seperable. By the lemma, there exists $h \in \mathcal{H}$, with $\|h\| = 1$ such that $[\mathcal{A}h]^- = \mathcal{H}$. Since $\mathcal{A}$ is an abelian C^* algebra $\mathcal{A} \cong C(X)$. Let $\pi : C(X) \to \mathcal{A}$, be the isomorphism such that $\mathcal{A} = \{\pi(f) : f \in C(X)\}$. Define $s : C(X) \to \mathbb{C}$ via $s(f) = \langle \pi(f)h, h \rangle$. Since s is a state there exists a measure μ such that $s(f) = \int_X f d\mu$.

For $f \in C(X) \subseteq L^2(X, \mu)$, set $Uf = \pi(f)h \in \mathcal{H}$.

$$\|Uf\|^2 = \|\pi(f)h\|^2 = \langle \pi(|f|^2)h, h \rangle = \int |f|^2 d\mu = \|f\|^2.$$

Since U is an isometry on $C(X)$, U extends to a map on the closure of C(X) in $L^2(X, \mu)$. Therefore, $U : L^2(X, \mu) \longrightarrow \mathcal{H}$ is an isometry.

To see that U is onto, note that $\{\pi(f)h : f \in C(X)\} = \{Ah : A \in \mathcal{A}\}$ which imples $\{UM_fU^* : f \in L^\infty(X, \mu)\} \supseteq \{UM_fU^* : f \in C(X)\} =$

$\{\pi(f) : f \in C(X)\} = \mathcal{A}$. However, $\{UM_fU^* : f \in L^\infty(X,\mu)\}$ is an abelian selfadjoint algebra which contains the MASA $\mathcal{A}$. Therefore, $\mathcal{A} = \{UM_fU^* : f \in L^\infty(X,\mu)\}$. □

Theorem 4.10 (MASA decomposition theorem)**.** *Let $\mathcal{H}$ be a seperable Hilbert space, and assume that $\mathcal{A} \subseteq B(\mathcal{H})$ is a MASA. Then there exists $\mathcal{H}_a \subseteq \mathcal{H}$ such that*

1) $\mathcal{P}_{\mathcal{H}_a} \subseteq \mathcal{A}$.
2) $\mathcal{A}_a = \{\mathcal{P}_{\mathcal{H}_a}A|_{\mathcal{H}_a} : A \in \mathcal{A}\} \subseteq B(\mathcal{H}_a)$ *is a discrete MASA.*
3) $\mathcal{A}_c = \{\mathcal{P}_{\mathcal{H}_a{}^\perp}A|_{\mathcal{H}_a{}^\perp} : A \in \mathcal{A}\} \subseteq B(\mathcal{H}_a^\perp)$ *is a continuous MASA.*

Once again we sketch the proof. We can assume without any loss of generality that $\mathcal{H} \subseteq L^2(X,\mu)$ and $\mathcal{A} = \{M_f : f \in L^\infty(X,\mu)\}$.
The rest follows from a decomposition theorem theorem for measures into **atomic** and **continuous** parts.

Given $x \in X, \{x\}$-Borel set. Let $X_a = \{x : \mu(x) \neq 0\}$. Write $X = X_a \cup X_c$ and argue that the fact that $\mathcal{H}$ is seperable forces X_a to be at most countable.

Let $\mu_a = \mu|_{X_a}, \mu_c = \mu|_{X_c}$ and $\mathcal{H}_a = span\{\delta_{\{x\}} : x \in X_a\}$.

Theorem 4.11 (Kadison-Singer)**.** *Let $\mathcal{A} \subseteq B(\mathcal{H})$ be a MASA. $\mathcal{H} = \mathcal{H}_a \oplus \mathcal{H}_c$. If $\mathcal{H}_c \neq 0$, then there exist a pure state on $\mathcal{A}$ which has a non unique state extensions to $B(\mathcal{H})$.*

Hence, the only case where could have uniqueness of extensions of pure states is when the MASA $\mathcal{A}$ is discrete.

4.1. **The Strong Operator Topology (SOT) and The Weak Operator topology(WOT).** Given a net $\{T_\lambda\}_{\lambda \in \Lambda} \subseteq B(\mathcal{H})$ we say that $T_\lambda \longrightarrow T$ in SOT iff $\|T_\lambda h - Th\| \to 0, \forall h \in \mathcal{H}$.

A net $T_\lambda \longrightarrow T$ in WOT iff $|\langle T_\lambda h, k\rangle - \langle Th, k\rangle| \to 0$, for all $h, k \in \mathcal{H}$.

Theorem 4.12 (von-Neumann Double Commutant Theorem)**.** *If $I \in \mathcal{A} \subseteq B(\mathcal{H})$ be a C^* algebra, then $\mathcal{A}'' = \mathcal{A}^{-SOT} = \mathcal{A}^{-WOT}$.*

Proposition 4.13. *Let $\mathcal{A} \subseteq B(\mathcal{H})$ be a discrete MASA, then every minimal projection is rank 1.*

Proof. Let $\{E_\lambda\}_{\lambda\in\Lambda} \subseteq \mathcal{A}$ be the set of minimal projections so that $\mathcal{A} = \{E_\lambda\}'_{\lambda\in\Lambda}$. If E_μ and E_λ are two minimal projections such that $E_\lambda \neq E_\mu$, then $E_\lambda E_\mu = E_\lambda E_\mu E_\lambda \leq E_\mu$. Also, $(E_\lambda E_\mu)^2 = E_\lambda E_\mu = (E_\lambda E_\mu)^*$. Therefore, $E_\lambda E_\mu$ is a projection in $\mathcal{A}$. Either $E_\lambda E_\mu = 0$ or $E_\lambda E_\mu = E_\mu$. However,

$$E_\mu = E_\lambda E_\mu = E_\mu E_\lambda E_\mu \leq E_\lambda$$

which contradicts the minimality of E_λ. Therefore, $E_\lambda E_\mu = 0$ for all minimal projections. Now fix a minimal projection E_μ and assume tha the rank of $E_\mu \neq 1$. This implies that $E_\mu = F + G$, where F and G are non-zero orthogonal projections. Now, $0 \leq E_\lambda F E_\lambda \leq E_\lambda E_\mu E_\lambda = 0$ for $\lambda \neq \mu$. Therefore, $E_\lambda F = 0$ for all $\lambda \neq \mu$. Similarly, $E_\lambda G = 0$, $FE_\lambda = 0$, $GE_\lambda = 0$ for all $\lambda \neq \mu$. Let $\mathcal{B}$ be the C^* algebra generated by I, $\{E_\lambda\} \cup \{F\} \cup \{G\}$.

The algebra $\mathcal{B}$ is abelian and $\mathcal{B}' \subseteq [\{E_\lambda\} \cup \{F\} \cup \{G\}]' \subseteq [\{E_\lambda\} \cup \{E_\mu\}]' = \mathcal{A}'$. Thus, $\mathcal{A} = \mathcal{A}'' \subseteq \mathcal{B}'' = \mathcal{B}^{-SOT}$. Let $T_1, T_2 \in \mathcal{B}^{-SOT}$, there exists $\{B_\lambda\}, \{C_\mu\} \subseteq \mathcal{B}$ such that $B_\lambda \longrightarrow T_1$ and $C_\mu \longrightarrow T_2$ in SOT. Now for any $B \in \mathcal{B}$,

$$T_1 B h = \lim_\lambda B_\lambda B h = \lim_\lambda B(B_\lambda h) = B T_1 h$$

Similarly $T_2 B = B T_2$ for all $B \in \mathcal{B}$. Finally,

$$T_1 T_2 h = \lim_\lambda B_\lambda T_2 h = \lim_\lambda T_2(B_\lambda h) = T_2 T_1 h.$$

Therefore $T_1, T_2 \in \mathcal{B}^{-SOT}$ implies $T_1 T_2 = T_2 T_1$. Hence $\mathcal{B}''$ is abelian. Since $\mathcal{A}$ is a MASA, $\mathcal{A} = \mathcal{B}''$. Now $F, G \in \mathcal{B}'' = \mathcal{A}$ contradicts the minimality of E_μ. Therefore, E_μ has rank 1. □

Theorem 4.14. *Let $\mathcal{H}$ be a separable, infinite dimensional Hilbert space and $\mathcal{A} \subseteq B(\mathcal{H})$ be a discrete MASA. There exists a unitary $U : \ell^2(\mathbb{N}) \to \mathcal{H}$ such that $\mathcal{A} = U\mathcal{D}U^* = \{UDU^* : D \in \mathcal{D}\}$, where $\mathcal{D} = \ell^\infty(\mathbb{N}) \subseteq B(\ell^2(\mathbb{N}))$ is the MASA of diagonal matrices.*

Proof. Let $\mathcal{A} = \{E_\lambda\}'_{\lambda \in \Lambda}$, where E_λ has rank 1. Choose $e_\lambda \in clH$ such that $\|e_\lambda\| = 1$, $E_\lambda e_\lambda = e_\lambda$ and $E_\lambda e_\mu = 0$ for $\lambda \neq \mu$. Thus $e_\lambda \perp e_\mu$ for $\lambda \neq \mu$ We claim that $\{e_\lambda\}_{\lambda \in \Lambda}$ is an orthonormal basis for $\mathcal{H}$. Assume $v \in \mathcal{H}$, $\|v\| = 1$, and $v \perp e_\lambda$, for all $\lambda \in \Lambda$. Let P be the rank one projection onto v. For all λ, $PE_\lambda = E_\lambda P = 0$. Hence, $P \in \{E_\lambda\}' = \mathcal{A}$, which implies that P is a minimal projection in $\mathcal{A}$, which is a contradiction. Therefore, $\{e_\lambda\}$ is an orthonormal basis for $\mathcal{H}$. The cardinality of Λ is given by $card(\Lambda) = dim(\mathcal{H})$ which implies that Λ is countable. Write $\Lambda = \{\lambda_n\}_{n \in \mathbb{N}}$ and define $U : \ell^2(\mathbb{N}) \to \mathcal{H}$ by $Ue_n = e_{\lambda_n}$. U is clearly a unitary. Since $\mathcal{D} \subseteq B(l^2(\mathbb{N}))$ is a MASA, $U\mathcal{D}U^* \subseteq B(\mathcal{H})$ is also a MASA. Now, $UE_{n,n}U^* = E_{\lambda_n}$ and so $E_{\lambda_n} \in U\mathcal{D}U^*$. Since $\{E_\lambda\}_{\lambda \in \Lambda} \subseteq U\mathcal{D}U^*$ we get $(U\mathcal{D}U^*)' \subseteq \{E_\lambda\}'_{\lambda \in \Lambda} \subseteq \mathcal{A}$ and so $U\mathcal{D}'U^* \subseteq \mathcal{A}$. The diagonal operators $\mathcal{D}$ are a MASA and so $\mathcal{D} = \mathcal{D}'$. It follows that $U\mathcal{D}U^* \subseteq \mathcal{A}$ and so $U\mathcal{D}U^* = \mathcal{A}$. □

The above result leads to an equivalent statement of the Kadison-Singer Problem.

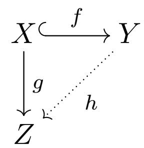
Kadison-Singer Problem. Does every pure state on $\ell^\infty(\mathbb{N}) = \mathcal{D}$ extend uniquely to a state on $B(\ell^2(\mathbb{N}))$?

5. The Stone-Cech Compactification

Let X be a locally compact Hausdorff space (LCH). A compactification of X is a pair (Y, f) where Y is a compact Hausdorff space (CH) and $f : X \to Y$ is a continuous function such that $f : X \to f(X)$ is a homeomorphism and $f(X)$ is dense in Y.

Theorem 5.1 (Stone-Cech)**.** *Let X be a locally compact Hausdorff space. Then there exist a compactification (Y, f) of X such that if Z is any compact*

Hausdorff space and $g : X \to Z$ is any continuous function, then there exists $h : Y \to Z$ continuous such that $h \circ f = g$.



We make note of a couple of facts about the Stone-Cech compactification.

(1) Since $f(X)$ is dense in Y, h is unique.
(2) If (Z, g) was another compactification of X, then since $g(X)$ is dense in Z, $h(Y) = Z$. In this sense, the Stone-Cech is the largest or maximal compactification of X.

Corollary 5.2. *If (W, j) is another compactification of X with the properties of (Y, f) then there exists a homeomorphism $h : Y \to W$ such that $h \circ f = j$, $h^{-1} \circ j = f$.*

There exists a map $f : X \to Y$ and $j : X \to W$ and so there exists h and g such that $h \circ f = j$ and $g \circ j = f$. Hence, $g \circ h(f(x)) = g((j(x)) = f(x)$ and so $g \circ h$ is the identity on a dense subset of Y so by continuity $g \circ h = id_Y$ and similarly $h \circ g = id_W$. So h is a homeomorphism with $g = h^{-1}$ and so $h \circ f = j$.

Definition 5.3. *The space (Y, f) is called the Stone-Cech compactification of X and we denote this βX.*

Let X be an LCH and recall $C_b(X)$ the space of bounded continuous functions on X. Given $f \in C(\beta X)$, $f : \beta X \to \mathbb{C}$ so $f \circ i : X \to \mathbb{C}$ is continuous and $f(X) \subset f(\beta X)$. The latter set is a compact subset of $\mathbb{C}$ and so is closed and bounded. Hence $f \circ i \in C_b(X)$.

Theorem 5.4. *The map $i^* : C(\beta X) \to C_b(X)$ given by $f \mapsto f \circ i$ is an (onto) $*$-isomorphism.*

Proof. $(f_1 + f_2) \circ i = f_1 \circ i + f_2 \circ i$ and so $i^*(f_1 + f_2) = i^*(f_1) + i^*(f_2)$. $i^*(f_1 f_2) = (f_1 f_2) \circ i = (f_1 \circ i)(f_2 \circ i) = i^*(f_1) i^*(f_2)$, $i^*(\overline{f}) = \overline{f} \circ i = \overline{f \circ i} = \overline{i^*(f)}$. Hence, i^* is a homomorphism.

To see that it is one-to-one assume that $i^*(f) = f \circ i = 0$ and so $f = 0$ on $i(X)$. But $i(X)$ is dense in βX and since f is continuous $f \equiv 0$.

Let $g \in C_b(X)$ and let $M := \sup\{|g(x)| : x \in X\}$. Let $Z = \{\lambda \in \mathbb{C} : |\lambda| \leq M\}$. $g : X \to Z$ and so by Stone-Cech there exists a function $\hat{g} : \beta X \to \mathbb{C}$ so that $\hat{g} \circ i = g$. But this just means $i^*(\hat{g}) = g$. □

Remark the map $g \mapsto \hat{g}$ maps $C_b(X) \to C(\beta X)$ is the inverse of i^*.

Recall the maximal ideal space of $C_b(X)$, which we denote $\mathcal{M}(C_b(X))$. We know this is the set of functions $\delta : C_b(X) \to \mathbb{C}$ such that δ is a non-zero, multiplicative linear functional.

We have seen that $\|\delta\| = 1$ and that the set of multiplicative linear functionals is a weak*-closed subset of the the ball of the dual space $C_b(X)_1^*$. Hence it is a weak*-compact Hausdorff space when endowed with the weak* topology.

Suppose that we are given a point $\omega \in \beta X$ then define $\delta_\omega : C_b(X) \to \mathbb{C}$ by $\delta_w(g) = \hat{g}(\omega)$. This gives a map $\Gamma : \beta X \to \mathcal{M}(C_b(X))$.

Proposition 5.5. $\Gamma : \beta X \to \mathcal{M}(C_b(X))$ *is a homeomorphism.*

Proof. If $\omega_\lambda \to \omega$ in βX this implies that $\hat{g}(\omega_\lambda) \to \hat{g}(\omega)$ for all $\hat{g} \in C(\beta X)$. It follows that $\delta_{\omega_\lambda}(g) \to \delta_\omega(g)$ for all $g \in C_b(X)$. Conversely if $\delta_{\omega_\lambda} \to \delta_\omega$ in the weak* topology then for all $\hat{g} \in C(\beta X)$ we have $\hat{g}(\omega_\lambda) \to \hat{g}(\omega)$. this implies that $\omega_\lambda \to \omega$. To see that it is onto let $\delta : C_b(X) \to \mathbb{C}$ and define $\tilde{\delta} : C(\beta X) \to \mathbb{C}$ by $\tilde{\delta}(\hat{g}) = \delta(g)$ and so $\tilde{\delta} \in \mathcal{M}(C(\beta X))$. It follows that there exists $\omega \in \beta X$ such that $\tilde{\delta} = \delta_\omega$. □

This shows we could define βX to be $\mathcal{M}(C_b(X))$ and then show that this has the universal property given in the Stone-Cech theorem. This is essentially Stone's proof.

An important example for us is the space $\ell^\infty(\mathbb{N})$. We can view this as the set of continuous bounded functions on $\mathbb{N}$ so $\ell^\infty(\mathbb{N}) = C_b(\mathbb{N}) = C(\beta\mathbb{N})$.

6. Ultrafilters

Ultrafilters will give us another approach to understanding $\beta\mathbb{N}$. Let S be a set. A non-empty collection $\mathcal{F}$ of non-empty subsets of S is called a filter provided

(1) If $n \geq 1$ and $F_1, \dots, F_n \in \mathcal{F}$, then $\bigcap_{j=1}^n F_j \in \mathcal{F}$.
(2) If $F \in \mathcal{F}$ and $F \subseteq G$, then $G \in \mathcal{F}$.

Note that the second property forces $S \in \mathcal{F}$. Note that if $F \in \mathcal{F}$, then $F \cap F^c = \emptyset$ and so $F^c \notin \mathcal{F}$. A filter is called an ultrafilter if it is not contained in any other filter, i.e. it is a maximal filter.

Therefore $\mathcal{U}$ is an ultrafilter if and only if $\mathcal{U}$ is a filter and if $\mathcal{F}$ is a such that filter $\mathcal{F} \supseteq \mathcal{U}$ then $\mathcal{F} = \mathcal{U}$.

Let $s_0 \in \mathcal{S}$ and let $\mathcal{U}_{s_0} := \{A \subseteq S : s_0 \in A\}$. It is straightforward to check that $\mathcal{U}_{s_0}$ is a filter. Now suppose that $\mathcal{F} \subseteq \mathcal{U}$ and that $\mathcal{U} \neq \mathcal{F}$. There exists $A \in \mathcal{F}$ such that $A \notin \mathcal{U}$. This implies $s_0 \notin A$. However, $\{s_0\} \in \mathcal{U}_{s_0} \subseteq \mathcal{F}$ and so $\emptyset = \{s_0\} \cap A \in \mathcal{F}$, which is a contradiction. Hence, $\mathcal{U}_{s_0}$ is an ultrafilter. The ultrafilters of the form $\mathcal{U}_{s_0}$ for some $s_0 \in S$ are called the **principal ultrafilters.**

Proposition 6.1. *Let S be a non empty set. Then a collection $\mathfrak{U}$ of subsets of S is an ultrafilter if and only if it is a filter and for each $A \subseteq S$ either $A \in \mathfrak{U}$ or $A^c \in \mathfrak{U}$.*

Proof. ($\Rightarrow$) Let $\mathrm{A} \subseteq \mathrm{S}$ and $\mathrm{A} \notin \mathfrak{U}$. We must show that $\mathrm{A}^c \in \mathfrak{U}$.
Let $\mathfrak{W} = \{ \mathrm{Y} \subseteq \mathrm{S} : \exists\, \mathrm{U} \in \mathfrak{U}, \mathrm{A}^c \cap \mathrm{U} \subseteq \mathrm{Y} \}$
For $\mathrm{Y}_1, \dots, \mathrm{Y}_n \in \mathfrak{W}$, there exits $\mathrm{U}_1, \dots, \mathrm{U}_n \in \mathfrak{U}$ such that $\mathrm{Y}_i \supseteq \mathrm{A}^c \cap \mathrm{U}_i$.

$\Rightarrow \cap_{i=1}^{n} \mathrm{Y}_i \supseteq \mathrm{A}^c \cap (\cap_{i=1}^{n} \mathrm{U}_i)$
$\Rightarrow \cap_{i=1}^{n} \mathrm{U}_i \in \mathfrak{W}$.
Therefore finite intersections of sets in $\mathfrak{W}$ is in $\mathfrak{W}$. Clearly, every superset of a set in $\mathfrak{W}$ is in $\mathfrak{W}$. If $\emptyset \in \mathfrak{W}$, then there is some U in $\mathfrak{U}$ such that $\emptyset \supseteq \mathrm{A}^c \cap \mathrm{U}$. This impies $\mathrm{A}^c \cap$ U is an empty set. Therefore U $\subseteq$ A and so A$\in \mathfrak{U}$ which is a contradiction. So, $\emptyset \notin \mathfrak{W}$. Hence $\mathfrak{W}$ is a filter. But U$\in \mathfrak{U}$, U$\supseteq$ $\mathrm{A}^c\cap$ U. This implies U$\in \mathfrak{W}$. Thus $\mathfrak{U} \subseteq \mathfrak{W}$. Therefore $\mathfrak{U} = \mathfrak{W}$. But $\mathrm{A}^c = \mathrm{A}^c \cap \mathrm{S}$ and S$\in \mathfrak{U}$. Therefore $\mathrm{A}^c \in \mathfrak{W} = \mathfrak{U}$.
($\Leftarrow$) Suppose $\mathfrak{U}$ is not an ultra filter. So, there exists a filter $\mathfrak{W}$ such that $\mathfrak{U} \subsetneq \mathfrak{W}$. This implies that there exists A in $\mathfrak{W}$, A$\notin \mathfrak{U}$. Then $\mathrm{A}^c \in \mathfrak{U} \subseteq \mathfrak{W}$. This gives $\emptyset = \mathrm{A}^c \cap$ A is in $\mathfrak{W}$ which is not possible. Hence $\mathfrak{U}$ is an ultrafilter. □

6.1. **Convergence along an ultrafilter.** Let S be a set, $\mathfrak{U}$ an ultrafilter on a set S, X a compact Hausdorff space. Assume we are given $x_s \in$ X for each s in S. We want to define $\lim_{\mathfrak{U}} x_s$.
For each set A $\in \mathfrak{U}$, let $C_A = \overline{\{x_s : s \in A\}} \subseteq X$

Theorem 6.2. *Let S, $\{x_s\}_{s\in S} \subseteq X$ and $\mathfrak{U}$ be as above. Then :*

1) *There exists $x_0 \in X$ such that $\bigcap_{A\in\mathfrak{U}} C_A = \{x_0\}$.*
2) *Given an open set V in X, $x_0 \in V$ there exists A in $\mathfrak{U}$ such that $x_0 \in C_A \subseteq V$.*

Proof. 1. Given $A_1, A_2 \in \mathfrak{U}$, $C_{A_1} \cap C_{A_2} \supseteq C_{A_1\cap A_2} \neq \emptyset$.
Therefore $C_{A_1} \cap \cdots \cap C_{A_n} \supseteq C_{A_1\cap\cdots\cap A_n} \neq \emptyset$ for each $n \geq 0$ and so, $\{C_A\}_{A\in\mathfrak{U}}$ has finite intersection property.
Therefore $\cap_{A\in\mathfrak{U}} C_A \neq \emptyset$
Assume there exists $x, y \in \bigcap_{A\in\mathfrak{U}} C_A, x \neq y$.
Pick open sets U, V such that $x \in U, y \in V$ and $U \cap V = \emptyset$. Let $B = \{s : x_s \in U\}$. Either $B \in \mathfrak{U}$ $or B^c \in \mathfrak{U}$. If $B \in \mathfrak{U}$, then $\bigcap_{A\in\mathfrak{U}} C_A \subseteq C_B \subseteq \overline{U} \subseteq V^c$.
This implies $y \notin \cap_{A\in\mathfrak{U}} C_A$, which is a contradiction.
Hence $B^c \in \mathfrak{U}$. But $B^c = \{s : x_s \notin U\}$ which implies $C_{B^c} \subseteq U^c$.
Therefore $x \notin C_{B^c}$ and thus $x \notin \cap_{A\in\mathfrak{U}} C_A$ which is a contradiction.
Thus $\cap_{A\in\mathfrak{U}} C_A$ cannot have more than one element. So, it has exactly one element.
2. For each $y \in V^c, y \notin \bigcap_{A\in\mathfrak{U}} C_A$. Then there exists $A_y \in \mathfrak{U}$ such that $y \notin C_{A_y}$ this implies $y \in C_{A_y}^c$. Thus $\{C_{A_y}^c : y \in V^c\}$ is an open cover of V^c. By compactness there exists $y_1, \ldots, y_n \in V^c$ such that $V^c = C_{A_{y_1}}^c \cup \cdots \cup C_{A_{y_n}}^c$
$\Rightarrow V \supseteq C_{A_{y_1}} \cap \cdots \cap C_{A_{y_n}} \supseteq C_{A_{y_1}\cap\cdots\cap A_{y_n}} = C_A$ for $A = A_{y_1} \cap \cdots \cap A_{y_n}$. □

Definition 6.3. *Given $S, \{x_s : s \in S\}, \mathfrak{U}$ as above, we call $\{x_0\} = \bigcap_{A\in\mathfrak{U}} C_A$ the limit along the ultrafilter and denote it by $x_0 = \lim_{\mathfrak{U}} x_s$*

Here is the simplest example of an ultrafilter limit. Let $s_0 \in S$ and let $\mathfrak{U}_{s_0} = \{A \subseteq S : \ s_0 \in A\}$ be the principal ultrafilter generated by s_0.. Note that $\{s_0\} \in \mathfrak{U}_{s_0}$ and $C_{s_0} = \overline{\{f(s_0)\}} = \{f(s_0)\}$. Hence $\lim_{\mathfrak{U}_{s_0}} f(s) = f(s_0)$.

Proposition 6.4. *Let S be a set, $\mathfrak{U}$ an ultrafilter, $\{x_s : s \in S\} \subseteq \mathbb{C}, \{y_s : s \in S\} \subseteq \mathbb{C}$ both bounded, $\alpha \in \mathbb{C}$. Then*
$\lim_{\mathfrak{U}} \alpha x_s = \alpha \lim_{\mathfrak{U}} x_s$, $\lim_{\mathfrak{U}}(x_s + y_s) = \lim_{\mathfrak{U}} x_s + \lim_{\mathfrak{U}} y_s$,
$\lim_{\mathfrak{U}} x_s y_s = (\lim_{\mathfrak{U}} x_s)(\lim_{\mathfrak{U}} y_s)$.

Proof. We will only do the product. Let $x = \lim \mathfrak{U} x_s, y = \lim_{\mathfrak{U}} y_s$. Given $\epsilon > 0$, pick $\delta > 0$ such that $|z - x| < \delta, |w - y| < \delta$ implies $|zw - xy| < \epsilon$.
Let $C_A = \overline{\{x_s : s \in A\}}, B_A = \overline{\{y_s : s \in A\}}, D_A = \overline{\{x_s y_s : s \in A\}}$.
Since $V = \{z : |z - x| < \delta\}, W = \{w : |w - y| < \delta\}$ are open therefore there exists $A_1, A_2 \in \mathfrak{U}$ such that $C_{A_1} \subseteq V, B_{A_2} \subseteq W$.
Look at $A_3 = A_1 \cap A_2 \in \mathfrak{U}$.

$$D_{A_3} \subseteq C_{A_3} B_{A_3} \subseteq C_{A_1} B_{A_2} \subseteq VW$$

This implies that $D_{A_3} \subseteq \{zw : |z - x| < \delta, |w - y| < \delta\} \subseteq \{\alpha : |\alpha - xy| < \epsilon\}$

$$\bigcap_{A \in \mathfrak{U}} D_A \subseteq \mathcal{B}(xy; \epsilon) \text{ for each } \epsilon > 0$$

Therefore $\bigcap_{A \in \mathfrak{U}} D_A = \{xy\}$. □

Lemma 6.5. *Let S be a set with discrete topology. Then every subset of S is open in βS.*

Proof. It is enough to show that for every $s_0 \in S$, $\{s_0\}$ is open in βS. Since the inclusion $i : S \to \beta S$ is a homeomorphism and $\{s_0\}$ is open in S, $\{s_0\}$ is relatively open in βS. So there exists an open set U in βS such that $U \cap S = \{s_0\}$. We claim that $U = \{s_0\}$. Indeed if not then the set $U - \{s_0\}$, which is necessarily open, will not be empty. But this implies that the elements of $U - \{s_0\}$ is not in the closure of S which is dense in βS. Contradiction. Hence $\{s_0\}$ is open in βS. □

Theorem 6.6. *Let S be a set with discrete topology. Then:*

1) *For every ultrafilter $\mathfrak{U}$ on S, there exists $w \in \beta S$ such that $\lim_{\mathfrak{U}} f(s) = \hat{f}(w)$ for all $f \in C_b(S)$.*
2) *If $\mathfrak{U}_1 \neq \mathfrak{U}_2$ are ultrafilters then $w_1 \neq w_2$ (one-to-one).*
3) *Given any $w \in \beta S$ there exists unique ultrafilter $\mathfrak{U}$ such that $\hat{f}(w) = \lim_{\mathfrak{U}} f(s)$ for all $f \in C_b(S)$.*

Proof. 1. Define $\delta_{\mathfrak{U}} : \mathcal{C}_b(S) \longrightarrow \mathbb{C}$ by $\delta_{\mathfrak{U}}(f) = \lim_{\mathfrak{U}} f(s)$. By the last proposition $\delta_{\mathfrak{U}}$ is a multiplicative linear functional. Therefore there exists $w \in \beta S$ such that

$$\lim_{\mathfrak{U}} f(s) = \delta_{\mathfrak{U}}(f) = \hat{f}(w).$$

2. Let $\mathfrak{U}_1 \neq \mathfrak{U}_2$ be two ultrafilters on S. By the maximality of ulatrafilters there exists $A \in \mathfrak{U}_1$ with $A \notin \mathfrak{U}_2$. So $A^c \in \mathfrak{U}_2$. Define $f : S \to \mathbb{C}$ by $f(s) = 1$ if $s \in A$ and $f(s) = 0$ if $s \notin A$. Clearly $f \in C_b(S)$ and we have that

$C_A = \overline{\{f(s) : \ s \in A\}} = \{1\}$ and $C_{A^c} = \{0\}$. Let w_1 and w_2 be two points in βS satisfying the conditions in part1 for $\mathfrak{U}_1$ and $\mathfrak{U}_2$, respectively. Then

$$1 = \lim_{\mathfrak{U}_1} = \hat{f}(w_1) \quad \text{and} \quad 0 = \lim_{\mathfrak{U}_2} = \hat{f}(w_2).$$

So $w_1 \neq w_2$.

3. Given $w \in \beta S$ let $\mathfrak{N}_w = \{U \subseteq \beta S : \ U \text{ is open }, w \in U\}$ and let $\mathfrak{U}_w = \{U \cap S : \ U \in \mathfrak{N}_w\}$. We claim that $\mathfrak{U}_w$ is an ultrafilter. First note that $U \cap S \neq \emptyset$ for all open sets U since S is dense. If $U_i \cap S \in \mathfrak{U}_w$ for $i = 1, 2, ..., n$ then

$$\bigcap_{i=1}^{n} (U_i \cap S) = (U_1 \cap U_2 \cap ... \cap U_n) \cap S \in \mathfrak{U}_w.$$

Let $U \cap S \in \mathfrak{U}_w$ and $U \cap S \subseteq B \subseteq S$. Let $V = U \cup \{B - (U \cap S)\}$. Then $V \in \mathfrak{N}_w$ and so $V \cap S = (U \cap S) \cup \{B - (U \cap S)\} = B$ is in $\mathfrak{U}_w$. This shows that $\mathfrak{U}_w$ is a filter. To see that $\mathfrak{U}_w$ is an ultrafilter let $A \subseteq S$. We claim that either A or A^c is in $\mathfrak{U}_w$. Define $f : S \to \{0, 1\} \subseteq \mathbb{R}$ by $f(s) = 0$ if $s \in A$ and $f(s) = 1$ if $s \in A^c$. Since f is bounded on S it extends uniquely to a continuous function $\hat{f} : \beta S \to \{0, 1\} \subseteq \mathbb{R}$. Let $U = \hat{f}^{-1}((-1/2, 1/2)) = \hat{f}^{-1}(-1)$ which must be both open and closed. Let $V = \hat{f}^{-1}(1)$. Then $V = U^c$ and both open and closed. Since $\hat{f}^{-1}(w) \in \{0, 1\}$ we get one of the following:

If $\hat{f}^{-1}(w) = 0 \ \Rightarrow \ w \in U \ \Rightarrow \ U \cap S = A \in \mathfrak{U}_w$ or

if $\hat{f}^{-1}(w) = 1 \ \Rightarrow \ w \in V \ \Rightarrow \ V \cap S = A^c \in \mathfrak{U}_w$.

So $\mathfrak{U}_w$ is an ultrafilter. Now we claim that $\lim_{\mathfrak{U}_w} f(s) = \hat{f}(w)$. Let $A \in \mathfrak{U}_w$. So $A = U \cap S$ for some open set U in βS containing w.

$$C_A = \overline{\{f(s) : \ s \in A\}} \subseteq \overline{\{f(w') : \ w' \in U\}} = \overline{\{f(U)\}}$$

$$\Longrightarrow \bigcap_{A \in \mathfrak{U}_w} C_A \subseteq \bigcap_{U \in \mathfrak{N}_w} \overline{\{f(U)\}} = \{\hat{f}(w)\}$$

$$\Longrightarrow \bigcap_{A \in \mathfrak{U}_w} C_A = \{\hat{f}(w)\}.$$

Hence we have $\hat{f}(w) = \lim_{\mathfrak{U}_w} f(s)$. □

Summary: There is a one-to-one correspondence between points in βS and the ultrafilters on S which can be described as

$$w \in \beta S \ \longleftrightarrow \ \mathfrak{U}_w = \{U \cap S : \ U \text{ is open in } \beta S \text{ and } w \in U\}.$$

Here is the simplest example of this correspondence. Let $s_0 \in S$, then since every subset of S is open, every subset of S that contains s_0 is an open neighborhood of s_0.. Hence, $\mathfrak{U}_{s_0} = \{U \cap S : U \in \mathcal{N}_{s_0}\} = \{A \subseteq S : \ s_0 \in A\}$, which is what we called the principal ultrafilter generated by s_0. Also, note that the earlier notation that we used for the principal ultrafilter is consistent with the notation that we are using above.

6.2. **Some Types of Ultrafilters.**

Definition 6.7. *Let N be a countable set and $\mathfrak{U}$ be an ultrafilter on N. Then $\mathfrak{U}$ is called*

selective *if for all partition $\{P_i\}_i$ of N either there is an i_0 such that $P_{i_0} \in \mathfrak{U}$ or there exits $B \in \mathfrak{U}$ such that card$(B \cap P_i) \leq 1$ for all i.*

rare *if for all partition $\{P_i\}_i$ of N into finite sets there exits $B \in \mathfrak{U}$ such that card$(B \cap P_i) \leq 1$ for all i.*

δ-stable *if for all partition $\{P_i\}_i$ of N either there is an i_0 such that $P_{i_0} \in \mathfrak{U}$ or there exits $B \in \mathfrak{U}$ such that card$(B \cap P_i) < +\infty$ for all i.*

Fact 1. selective $\Longleftrightarrow$ rare and δ-stable.

Recall that continium hypothesis says there is no set with cardinality greater than card($\mathbb{N}$) and less than card($\mathbb{R}$).

Fact 2. These ultrafilters not known to exists without assuming continium hypothesis. But under the assumtion of continium hypothesis these there types not only exist but also form a dense subset in $\beta N - N$.

Recall that a subset of a topological space is called a G_δ set if it can be written as intersection of open sets.

Definition 6.8. *Let X be a compact Hausdorff space. A point $x_0 \in X$ is called a **P-point** if every G_δ set containing x_0 contains a neigborhood of x_0.*

Remark. If $N^* = \beta N - N$ is a compact Hausdorff space then an element w of N^* is a P-point if and only if $\mathfrak{U}_w$ is δ-state.

6.3. **Some Topological Properties of βS when S is a Discrete Space.**

Let S be a discrete topological space, $A \subseteq S$ and $B = S \backslash A$, so that $S = A \amalg B$ a disjoint union. Under this context we would like to see what the relationship is between the closure of A and the closure of B in βS.

Proposition 6.9. *If $\chi_A : S \to \{0,1\}$ is the characteristic function of A and $h = \hat{\chi_A} : \beta S \to \{0,1\}$ its unique extension to the Stone-Cech compactification of S, then*

$$\begin{aligned} \{\omega \in \beta S \ ; \ h(w) = 1\} &= \overline{A} \text{ in } \beta S, \text{ and} \\ \{\omega \in \beta S \ ; \ h(w) = 0\} &= \overline{B} \text{ in } \beta S. \end{aligned}$$

Moreover, $\overline{A} \cap \overline{B} = \emptyset$, $\overline{A} \cup \overline{B} = \beta S$, and $h = \chi_{\overline{A}}$.

Proof. From the point set topology we have $\overline{A} \cup \overline{B} = \overline{A \cup B}$, but $\overline{A \cup B} = \overline{S} = \beta S$ by the fact that S is dense in βS.

Now S is discrete, so the map χ_A, hence h, is continuous, thus

$$\begin{aligned} h(A) = \{1\} &\implies h(\overline{A}) = 1, \text{ and similarly} \\ h(B) = \{0\} &\implies h(\overline{B}) = 0. \end{aligned}$$

Therefore, we must have that $\overline{A} \cap \overline{B} = \emptyset$. From this it follows that $h = \chi_{\overline{A}}$. □

Note that $\beta S = \overline{A} \cup \overline{B}$ and $\overline{A} \cap \overline{B} = \emptyset$ ensure that both $\overline{A}$ and $\overline{B}$ are open and closed sets. Recall, a set is called **clopen** if it is both open and closed.

With ultraflters in mind, we want to use these facts to sort out a bit more carefully what happens when we take open sets in βS and intersect them with S.

Proposition 6.10. *Let S be a discrete space, $U \subseteq \beta S$ an open subset, and set $A = U \cap S$. Then $\overline{U} = \overline{A}$, hence $\overline{U}$ is clopen. Furthermore, $\overline{U} \cap S = A = U \cap S$.*

Proof. Well, $A \subseteq U$, so $\overline{A} \subseteq \overline{U}$. As for the reverse inclusion, let $\omega \in \overline{U}$, and take any open neighborhood of omega $V \in \mathcal{N}_\omega$. By the definition of closure, $V \cap U \neq \emptyset$, and since S is dense, $V \cap U \cap S \neq \emptyset$. That is, $V \cap A \neq \emptyset$. Since V was arbitrary, $\omega \in \overline{A}$, whence $\overline{U} = \overline{A}$.
Denoting as before the complement of A in S by $B = S \backslash A$, the previous proposition guarantees that $\overline{A} \cap \overline{B} = \emptyset$. Subsequently,

$$\overline{U} \cap \overline{B} = \overline{A} \cap \overline{B} = \emptyset \Longrightarrow \overline{U} \cap B = \emptyset \Longrightarrow \overline{U} \cap S = A.$$

□

In the above context, notice that the closure of any open set in βS is actually open too. That is quite remarkable topologically. Also, one would think that $\overline{U} \cap S \supsetneq U \cap S$, but in fact we have equality here, that is, no points are being added by closing the open set and then intersecting with S.

The preceding two propositions grant the following corollary.

Corollary 6.11. *Let S be a discrete space, $U \subseteq \beta S$ an open subset, and $A = U \cap S$. Then $\hat{\chi_A} = \chi_{\overline{U}}$.*

Proof. By the propositions, $\hat{\chi_A} = \chi_{\overline{A}} = \chi_{\overline{U}}$. □

Definition 6.12. *A topological space is called **extremally disconnected**, or **Stonian**, if the closure of every open set is open.*

The Stone-Cech compactification of a discrete space is Stonian. The literature uses the word *extremally* to distinguish from the adverb *extremely* which is often used interchangeably with *very*. So one may say that the Cantor set is *extremely* disconnected but it is not *extremally* disconnected in the above sense.

7. Paving Theory

In this section we derive Anderson's[**?**], [**?**] paving results and many related results inspired by his work. Our approach is a little different from Anderson's and is partially inspired by Hadwin's lecture notes [**?**] and the recent paper [**?**]

Definition 7.1. *Let $\mathcal{A}$ be a unital C^*-algebra, and $\mathcal{S} \subseteq \mathcal{A}$ be a subspace.*

1. *$\mathcal{S}$ is called an **operator system** if it contains the unit e_A, and is closed under involution, that means $x \in \mathcal{S} \Longrightarrow x^* \in \mathcal{S}$.*
2. *If $\mathcal{B}$ is C^*-algebra, a linear map $\phi : \mathcal{S} \to \mathcal{B}$ is said to be positive if $0 \leq p \in \mathcal{S}$ ensures $\phi(p) \geq 0$ in $\mathcal{B}$.*

Notice at once that the properties of $\mathcal{S}$ ensure that in writing the cartesian decomposition

$$x = \underbrace{\frac{1}{2}(x + x^*)}_{u} + i\,\underbrace{\frac{1}{2}i(x^* - x)}_{v}$$

of an element x in $\mathcal{S}$, both u, v belong to the operator system $\mathcal{S}$. However, when looking at the orthogonal decomposition $u = p - q$ of a hermitian $u \in \Re(\mathcal{S})$, p and q both belong to the norm closed $*$-algebra generated by x, namely $C^*(a) = \mathrm{cl}(\{p(x, x^*)\ ;\ p \in \mathbb{C}[X, Y]\})$, but are not necessarily members of the operator system. Regardless, we can still write x as the difference of two positive elements belonging to $\mathcal{S}$. Indeed,

$$x = \underbrace{\frac{1}{2}(\|x\|e + x)}_{p_1} - \underbrace{\frac{1}{2}(\|x\|e - x)}_{p_2}.$$

Here are a few facts about operator systems and positive maps.

Proposition 7.2. *Let $\mathcal{S}$ be an operator system, and $s : \mathcal{S} \to \mathbb{C}$ a positive linear functional.*

1. *If $h = h^* \in \mathcal{S}$, then $s(h) \in \mathbb{R}$.*
2. *For every $x \in \mathcal{S}$, $s(x^*) = \overline{s(x)}$.*

Proof. Write $h = p - q$ where p and q are positive elements in $\mathcal{S}$, then $s(h) = s(p) - s(q) \in \mathbb{R}$ because both $s(p)$ and $s(q)$ are real.
For $x \in \mathcal{S}$, write $x = u + iv$ where $u, v \in \Re(\mathcal{S})$. Then since $s(u)$ and $s(v)$ are real numbers,

$$s(x^*) = s(u - iv) = s(u) - is(v) = \overline{s(u) + is(v)} = \overline{s(u + iv)} = \overline{s(x)}.$$

□

Proposition 7.3. *Let $\mathcal{S} \subseteq \mathcal{A}$ be an operator system, and $s : \mathcal{S} \to \mathbb{C}$ a linear map with $s(e) = 1$. Then*

$$s \text{ is positive} \iff \|s\| = 1.$$

Proof. ($\Rightarrow$): Let $x \in \mathcal{S}$ with $\|x\| = 1$. If $s(x) = \lambda$, pick $|\omega| = 1$ such that $\omega\lambda = |\lambda|$. Then $s(\omega x) = |\lambda|$. Also $\|\omega x\| = 1$ which gives $\|\omega x + (\omega x)^*\| \leq 2$. Using the fact that s is Hermitian and considering spectra, we get

$$\begin{aligned} -2e \le \omega x + (\omega x)^* \le 2e \quad &\Rightarrow \quad s(-2e) \le s(\omega x + (\omega x)^*) \le s(2e) \Rightarrow \\ -2 \le s(\omega x) + \overline{s(\omega x)} \le 2 \quad &\Rightarrow \quad -2 \le |\lambda| + |\lambda| \le 2 \Rightarrow |\lambda| \le 1. \end{aligned}$$

Therefore $|s(x)| \le 1$ for every $\|x\| = 1$. At the same time $s(e) = 1$, hence $\|s\| = 1$.
($\Leftarrow$): The argument in Proposition **??** works perfectly well here. □

Definition 7.4. *A positive linear map $s : \mathcal{S} \longrightarrow \mathbb{C}$ defined on an operator system $\mathcal{S}$ with $s(e) = 1$ is called a **state**.*

Corollary 7.5. *If $\mathcal{S} \subseteq \mathcal{A}$ is an operator system, every state on $\mathcal{S}$ extends to a state on $\mathcal{A}$*

Proof. If $s : \mathcal{S} \longrightarrow \mathbb{C}$ is a state, then $\|s\| = 1$, so we can employ the Hahn-Banach theorem to get $\tilde{s} : \mathcal{A} \longrightarrow \mathbb{C}$ with $\|\tilde{s}\| = 1$ and $\tilde{s}|_{\mathcal{S}} = s$. Then $\tilde{s}(e) = s(e) = 1$, so $\tilde{s}$ is also a state. □

The existence of the extension having been shown, let us investigate the uniqueness. To that end we give the following definition.

Definition 7.6. *Let $\mathcal{S} \subseteq \mathcal{A}$ be an operator system, and $s : \mathcal{S} \to \mathbb{C}$ a state. Set*

$$\mathcal{C}_s = \{\tilde{s} : \mathcal{A} \to \mathbb{C} \ ; \ \tilde{s}|_{\mathcal{S}} = s, \ \tilde{s} \text{ positive } \},$$

and let

$$\mathcal{U}(s) = \{x \in \mathcal{A} \ ; \ \tilde{s}_1(x) = \tilde{s}_2(x) \ \forall \tilde{s}_1, \tilde{s}_2 \in \mathcal{C}_s\}.$$

*$\mathcal{U}(s)$ is called **the uniqueness domain of** s.*

It is not to hard to see that $\mathcal{C}_s$ is weak*-closed and convex.

Proposition 7.7. *Let $\mathcal{S} \subseteq \mathcal{A}$ be an operator system, s a state on $\mathcal{S}$,*

1. *$\mathcal{S} \subseteq \mathcal{U}(s)$, and $\mathcal{U}(s)$ is an operator system.*
2. *If $x = u + iv \in \mathcal{A}$, then $x \in \mathcal{U}(s) \Longleftrightarrow u, v \in \mathcal{U}(s)$*

Proof.
(1) Clearly $\mathcal{S} \subseteq \mathcal{U}(s)$, and $\mathcal{U}(s)$ is a subspace. Let $\tilde{s}_1, \tilde{s}_2 \in \mathcal{C}_s$, then

$$\tilde{s}_1(x) = \tilde{s}_2(x) \ \Rightarrow \ \overline{\tilde{s}_1(x)} = \overline{\tilde{s}_2(x)} \Rightarrow \ \tilde{s}_1(x^*) = \tilde{s}_2(x^*),$$

therefore x^* belongs to $\mathcal{U}(s)$, and the latter is an operator system.
(2) Since $\mathcal{U}(s)$ is an operator system, $x = u + iv \in \mathcal{U}(s)$ implies that $u, v \in \mathcal{U}(s)$. The opposite direction relies on the fact that $\mathcal{U}(s)$ is a subspace.

□

Definition 7.8. *Let $\mathcal{S} \subseteq \mathcal{A}$ be an operator system, $s : \mathcal{S} \to \mathbb{C}$ a state, and $k = k^* \in \mathcal{A}$. We set*

$$\begin{aligned} \ell_s(k) &= \sup_{h \in \mathcal{S}, h \le k} s(h), \\ u_s(k) &= \inf_{h \in \mathcal{S}, k \le h} s(h). \end{aligned}$$

the ***respective lower envelope and upper envelope of*** *$k \in \mathcal{A}$.*

Theorem 7.9. *Let $\mathcal{S} \subseteq \mathcal{A}$ be an operator system, $s : \mathcal{S} \to \mathbb{C}$ a state, and $k = k^* \in \mathcal{A}$. Then*

1. *$\ell_s(k) \le u_s(k)$.*
2. *For any $t \in [\ell_s(k), u_s(k)]$, there is a state $s_t \in \mathcal{C}_s$ such that $s_t(k) = t$.*

Proof. If $k \in \mathcal{S}$ everything is clear, so assume that k does not belong to $\mathcal{S}$. Well if h and h' are real in $\mathcal{S}$ with $h \le k \le h'$ then $s(h) \le s(h')$. Fixing $s(h')$ and taking supremums of all such $s(h)$, and then taking infimums of all such $s(h')$ yields the desired inequality.

Let $\mathcal{S}_1 = \text{lin span}\{\mathcal{S}, k\}$. This is indeed an operator system. Now define a linear map $f : \mathcal{S}_1 \longrightarrow \mathbb{C}$ as

$$f(x + \alpha k) = s(x) + \alpha t, \ x \in \mathcal{S}.$$

We claim that f is a state on $\mathcal{S}_1$. Well $f(e) = 1$. Now let's show that f is positive. To that end, if $x + \alpha k \ge 0$, then

$$x^* + \overline{\alpha} k = x^* + \overline{\alpha} k^* = (x + \alpha k)^* = x + \alpha k.$$

Therefore, $x = x^*$, and $\alpha = \overline{\alpha}$ is real. There are three cases.

- $\alpha = 0$. Then $x \ge 0$, so $f(x) = s(x) \ge 0$.
- $\alpha > 0$.

$$\begin{aligned} x + \alpha k \ge 0 \quad &\Rightarrow \quad \alpha k \ge -x \ \Rightarrow \ k \ge -\alpha^{-1} x \ \Rightarrow \ s(k) \ge s(-\alpha^{-1} x) \\ &\Rightarrow \quad t \ge -\alpha^{-1} s(x) \ \Rightarrow \ f(x + \alpha k) = s(x) + \alpha t \ge 0. \end{aligned}$$

- $\alpha < 0$. Then

$$\begin{aligned} x + \alpha k \ge 0 \quad &\Rightarrow \quad x \ge -\alpha k \ \Rightarrow \ -\alpha^{-1} x \ge k \ \Rightarrow \ s(-\alpha^{-1} x) \ge s(k) = t \\ &\Rightarrow \quad f(x + \alpha k) = s(x) + \alpha t \ge 0. \end{aligned}$$

Therefore, f is a state on $\mathcal{S}_1$. Now we proved that we can always extend a state to the whole of $\mathcal{A}$, so extend f to $\tilde{f} : \mathcal{A} \to \mathbb{C}$, and set $\tilde{f} = s_t$.

□

Corollary 7.10. *Let $\mathcal{S} \subseteq \mathcal{A}$ be an operator system, $s : \mathcal{S} \to \mathbb{C}$ a state, and $k = k^* \in \mathcal{A}$. Then*

$$k \in \mathcal{U}(s) \Longleftrightarrow \ell_s(k) = u_s(k).$$

Remark: Conversely, if $\tilde{s} : \mathcal{A} \to \mathbb{C}$ is any state extending $s : \mathcal{S} \to \mathbb{C}$, then $l_s(h) \leq \tilde{s}(h) \leq u_s(h)$. This interval exactly characterizes the set of values of extensions of s at h.

Corollary 7.11. *Let $\mathcal{S} \subseteq \mathcal{A}$ be an operator system, $s : \mathcal{S} \to \mathbb{C}$ any state. Then $x \in \mathcal{U}(s)$ if and only if $l_s(Re(x)) = u_s(Re(x))$ and $l_s(Im(x)) = u_s(Im(x))$.*

For the following corollary, let $a, b \in \mathbb{R}, a \leq b$, denote $\mathcal{H}[a,b] = \{h = h^* \in \mathcal{A} : a \cdot e \leq h \leq b \cdot e\}$.

Corollary 7.12. *Let $\mathcal{S} \subseteq \mathcal{A}$ be an operator system, $s : \mathcal{S} \to \mathbb{C}$ any state, let $a, b \in \mathbb{R}, a \leq b$. Then, the followings are equivalent:*

1) *s extends uniquely to a state in $\mathcal{A}$,*
2) *$\forall h = h^* \in \mathcal{A}$, $l_s(h) = u_s(h)$,*
3) *$\forall h \in \mathcal{H}[a,b]$, $l_s(h) = u_s(h)$.*

Proof. (1) $\Rightarrow$ (2) : Assuming (1) is true, we get $\mathcal{U}(s) = \mathcal{A}$, which implies (2), using the last corollary.
(2) $\Rightarrow$ (3) is trivial.
(3) $\Rightarrow$ (1) : Assume (3) is true, then given any $h = h^* \in \mathcal{A}$, there exist $\alpha, \beta \in \mathbb{R}, \alpha \neq 0$, such that $\alpha h + \beta e \in \mathcal{H}[a,b]$, which implies $l_s(\alpha h + \beta e) = u_s(\alpha h + \beta e) \Rightarrow \alpha h + \beta e \in \mathcal{U}(s)$. So, $h = \dfrac{\alpha h + \beta e - \beta e}{\alpha} \in \mathcal{U}(s)$, i.e. every self-adjoint element is in $\mathcal{U}(s) \Rightarrow \mathcal{U}(s) = \mathcal{A}$. Done. □

Example 5 Let $\mathcal{A} = l^\infty([0,1])$, the bounded functions on $[0,1]$, let $\mathcal{S} = C([0,1]) \subseteq \mathcal{A}$ be an operator system(easy to show). Define $s : \mathcal{S} \to \mathbb{C}$ by $s(f) = \int_0^1 f(t)dt$, the Riemann integral, which is a state. Let $g = g^* \in \mathcal{A}$ be a real-valued bounded function. Then

$$l_s(g) = \sup\{\int_0^1 f(t)dt : f \leq g, f \in C([0,1])\} = \underline{\int_0^1} g(t)dt, \text{ **lower Riemann integral**},$$

and

$$u_s(g) = \inf\{\int_0^1 f(t)dt : g \leq f, f \in C([0,1])\} = \overline{\int_0^1} g(t)dt, \text{ **upper Riemann integral**}.$$

Therefore $\mathcal{U}(s)$ is the set of Riemann Integrable functions.
Apply these ideas to the Kadison-Singer case: Say $\mathcal{S} = \mathcal{D} = l^\infty(\mathbb{N}) = C(\beta N) \subseteq B(l^2(\mathbb{N})) = \mathcal{A}$, the state $s_w : \mathcal{D} \to \mathbb{C}$ as $s_w(D) = f_D(w)$, the point evaluation function, which is in fact, a pure state.

Lemma 7.13 (PR)**.** *Let $\mathcal{H}, \mathcal{K}$ be any two Hilbert spaces, $H = H^* = \begin{bmatrix} A & B \\ B^* & C \end{bmatrix} \in B(\mathcal{H} \oplus \mathcal{K})$, with A positive and invertible. Then there exists $\delta \geq 0$, such that $H + \delta P_{\mathcal{K}} = \begin{bmatrix} A & B \\ B^* & C + \delta I_{\mathcal{K}} \end{bmatrix}$ is positive.*

Proof. Let $X = A^{-1/2}B$, then for any $h \in \mathcal{H}, k \in \mathcal{K}$,

$$\begin{aligned}\langle \begin{bmatrix} A & B \\ B^* & C+\delta I_{\mathcal{K}} \end{bmatrix} \begin{pmatrix} h \\ k \end{pmatrix}, \begin{pmatrix} h \\ k \end{pmatrix} \rangle &= \langle \begin{pmatrix} Ah+Bk \\ B^*h+(C+\delta)k \end{pmatrix}, \begin{pmatrix} h \\ k \end{pmatrix} \rangle \\ &= \langle Ah,h\rangle + \langle Bk,h\rangle + \langle B^*h,k\rangle + \langle Ck,k\rangle + \delta\|k\|^2 \\ &= \langle A^{1/2}h, A^{1/2}h\rangle + \langle A^{1/2}Xk,h\rangle + \langle X^*A^{1/2}h,k\rangle + \langle Ck,k\rangle + \delta\|k\|^2 \\ &\geq \|A^{1/2}h\|^2 - 2\|Xk\|\|A^{1/2}h\| - \|C\|\|k\|^2 + \delta\|k\|^2 \\ &\geq (\|A^{1/2}h\| - \|Xk\|)^2 + \|k\|^2(\delta - \|C\| - \|X\|^2) \geq 0,\end{aligned}$$

whenever $\delta \geq \|C\| + \|X\|^2$. □

Theorem 7.14 (Kadison-Singer)**.** *Let $n \in \mathbb{N}$, and let $s_n : \mathcal{D} \to \mathbb{C}$, given by $s_n(D) = d_{nn}$, be a pure state. Then s_n extends uniquely to a state on $\mathcal{A} = B(l^2(\mathbb{N}))$.*

Proof. Using the lemma above, it's enough to show that for all $H = H^* = (h_{ij}) \in B(l^2(\mathbb{N}))$, we have $l_{s_n}(H) = u_{s_n}(H) = h_{nn}$.
Let $\mathcal{H} = span\{e_n\}$, $\mathcal{K} = \mathcal{H}^{\perp}$, then $\mathcal{H} \oplus \mathcal{K} = l^2(\mathbb{N})$. For any $\epsilon \geq 0$, look at $(h_{nn} - \epsilon)E_{nn}$, then we have $H - (h_{nn} - \epsilon)E_{nn} = \begin{bmatrix} \epsilon & * \\ * & * \end{bmatrix}$ in $\mathcal{H} \oplus \mathcal{K}$.
Therefore, $\exists\, \delta \geq 0$ such that $H - (h_{nn} - \epsilon)E_{nn} + \delta P_{\mathcal{K}} \geq 0$. This implies $H \geq (h_{nn} - \epsilon)E_{nn} - \delta P_{\mathcal{K}} \in \mathcal{D}$, where $P_{\mathcal{K}} = I_{\mathcal{K}} - E_{nn}$.
So, $l_s(H) \geq s_n((h_{nn} - \epsilon)E_{nn} - \delta P_{\mathcal{K}}) = h_{nn} - \epsilon \Rightarrow h_{nn} \leq l_{s_n}(H) \leq u_{s_n}(H)$.
Similarly, we show that $H \leq (h_{nn} + \epsilon)E_{nn} + \delta' P_{\mathcal{K}}$, which gives $u_{s_n}(H) \leq h_{nn} + \epsilon$, i.e. $u_{s_n}(H) \leq h_{nn}$. Hence, $l_{s_n}(H) = u_{s_n}(H) = h_{nn}$. □

Lemma 7.15. *Let $s : \mathcal{A} \to \mathbb{C}$ be a state, $P = P^* = P^2 \in \mathcal{A}$. If $s(P) = 1$, then $\forall X \in \mathcal{A}$, $s(PXP) = s(PX) = s(XP) = s(X)$.*

Proof. Using GNS, we have $s(Y) = \langle \pi(Y)\eta, \eta\rangle$, where $\pi : \mathcal{A} \to B(\mathcal{H}_\pi)$ is a *-homomorphism, and $\|\eta\| = 1$. Then $1 = s(P) = \langle \pi(P)\eta, \eta\rangle$ and $\pi(P)$ is a projection. Decompose $\eta = \underbrace{\pi(P)\eta}_{\eta_1} + \underbrace{(I - \pi(P))\eta}_{\eta_2} = \eta_1 + \eta_2$, then $1 = \langle \pi(P)\eta, \eta\rangle = \langle \eta_1, \eta_1 + \eta_2\rangle = \langle \eta_1, \eta_1\rangle = \|\eta_1\|^2$, which implies $\eta_2 = 0$ and $\pi(P)\eta = \eta$. Therefore, $s(XP) = \langle \pi(XP)\eta, \eta\rangle = \langle \pi(X)\underbrace{\pi(P)\eta}_{\eta}, \eta\rangle = \langle \pi(X)\eta, \eta\rangle = s(X)$. The rest is similar. □

Definition 7.16. *Given $A \subseteq \mathbb{N}$, we define $P_A = diag(d_{ii})$, $d_{ii} = \chi_A(i)$.*

Theorem 7.17. *Let $\omega \in \beta\mathbb{N}$, $s_w : \mathcal{D} \to \mathbb{C}$ be the *-homomorphism given by evaluation at ω and let $H^* = H \in B(l^2(\mathbb{N}))$. Then $l_{s_\omega}(H) = u_{s_\omega}(H) = t$ if and only if for every $\epsilon > 0$ there exists $A \in \mathfrak{U}(\omega)$ such that $(t - \epsilon)P_A \leq P_A H P_A \leq (t + \epsilon)P_A$.*

Proof. Let $s : \mathcal{D} \to \mathbb{C}$ be any state that extends s_w. If $U \in \mathfrak{N}_\omega$ and $A = U \cap \mathbb{N}$ then $P_A \in \mathcal{D}$ corresponds to $\chi_{\bar{A}} = \chi_{\bar{U}} \in C(\beta\mathbb{N})$. Note that

$s(P_A) = s_\omega(P_A) = \chi_{\bar{A}} = 1$. Thus, if the second condition holds, then $t-\epsilon = s((t-\epsilon)P_A) \leq s(P_AHP_A) = s(H) \leq t+\epsilon$ and hence, $s(H) = t$ for all s that extends s_ω. Thus $l_{s_\omega}(H) = u_{s_\omega}(H) = t$.
Conversely, if the first condition holds, then given $\epsilon > 0$ there exists $\mathcal{D}_1, \mathcal{D}_2 \in \mathcal{D}$ with $\mathcal{D}_1 \leq H \leq \mathcal{D}_2$ such that $(t - \frac{\epsilon}{2})P_A \leq s_w(\mathcal{D}_1) \leq s_w(\mathcal{D}_2) \leq (t+\frac{\epsilon}{2})P_A$. Now $\mathcal{D}_i \in P_A$ corresponds to a function $f_i \in C(\beta\mathbb{N})$, $f_1 \leq f_2$ since $\mathcal{D}_1 \leq \mathcal{D}_2$. Therefore, $s_w(\mathcal{D}_i) = f_i(w)$ and hence $(t-\frac{\epsilon}{2}) \leq f_1(\omega) \leq f_2(\omega) \leq (t+\frac{\epsilon}{2})$. *Pick* $U \in \mathfrak{N}_\omega$ such that for all $w' \in U$, $(t-\frac{\epsilon}{2}) \leq f_1(w') \leq f_2(w') \leq (t+\frac{\epsilon}{2})$. Let $A = U \cap \mathbb{N}$, P_A corresponds to $\chi_{\bar{U}}$ and thus, $P_A\mathcal{D}_1P_A$ corresponds to $\chi_{\bar{U}} f_1 \chi_{\bar{U}} \geq (t-\epsilon)\chi_{\bar{U}} \Rightarrow P_A\mathcal{D}_1P_A \geq (t-\epsilon)P_A$. Similarly, $P_A\mathcal{D}_2P_A \leq (t+\epsilon)P_A$ which implies $(t-\epsilon)P_A \leq P_A\mathcal{D}_1P_A \leq P_AHP_A \leq P_A\mathcal{D}_2P_A \leq (t+\epsilon)P_A$. This proves the theorem. □

Theorem 7.18. *Let* $H = H^* \in B(l^2(\mathbb{N}))$. *Then every pure state on* $\mathcal{D}$ *extends uniquely to H, i.e.* $H \in \mathfrak{U}(w) \forall w \in \beta\mathcal{N}$ *if and only if for each* $\epsilon > 0$, $\exists$ *finite collection of disjoint sets,* $B_1, B_2, ..., B_k$ *with* $B_1 \cup B_2 \cup ... \cup B_k = \mathbb{N}$ *and* $t_1, t_2, ..., t_k \in \mathbb{R}$ *such that* $(t_i - \epsilon)P_{B_i} \leq P_{B_i}HP_{B_i} \leq (t_i+\epsilon)P_{B_i}$.

Proof. Suppose the first condition holds true then for each $w \in \beta\mathbb{N}$ there exists $A_w \in \mathfrak{U}_w$ and $t_w \in \mathbb{R}$ such that $(t_w - \epsilon)P_{A_w} \leq P_{A_w}HP_{A_w} \leq (t_w+\epsilon)P_{A_w}$, where $A_w = U_w \cup \mathbb{N}$, $U_w \in \eta_w$. The collection $\{U_w : w \in \beta\mathbb{N}\}$ of open sets in $\beta\mathbb{N}$ covers the compact space $\beta\mathbb{N}$, therefore, we can choose $U_{w_1}, U_{w_2}, ..., U_{w_n}$ such that $\beta\mathbb{N} \subseteq U_{w_1} \cup U_{w_2} \cup ... \cup U_{w_n}$ which implies $\mathbb{N} \subseteq A_{w_1} \cup ... \cup A_{w_n}$. From here we may pick finitely many disjoint sets $B_1, B_2, ..., B_k$ such that $\mathbb{N} = B_1 \cup ... \cup B_k$ and each $B_i \in A_{w_i}$. For each i ,$1 \leq i \leq k$, pick w_l such that $B_i \subseteq A_{w_l}$ and set $t_i = t_{w_l}$. Then $(t_{w_l} - \epsilon)P_{A_{w_l}} \leq P_{A_{w_l}}HP_{A_{w_l}} \leq (t_{w_l}+\epsilon)P_{A_{w_l}}$ $\Rightarrow (t_{w_l} - \epsilon)P_{B_i}P_{A_{w_l}}P_{B_i} \leq P_{B_i}P_{A_{w_l}}HP_{A_{w_l}}P_{B_i} \leq (t_{w_l}+\epsilon)P_{B_i}P_{A_{w_l}}P_{B_i}$ which by using $P_{B_i}P_{A_{w_l}}P_{B_i} = P_{B_i}$ and $t_i = t_{w_l}$ gives the required inequality.
Conversely, fix w, let $\epsilon > 0$, then there exists finite collection of disjoint sets, $B_1, B_2, ..., B_k$ with $B_1 \cup B_2 \cup ... \cup B_k = \mathbb{N}$ and $t_1, t_2, ..., t_k \in \mathbb{R}$ such that $(t_i - \epsilon)P_{B_i} \leq P_{B_i}HP_{B_i} \leq (t_i+\epsilon)P_{B_i}$.
We claim that $\exists$ i such that $B_i \in \mathfrak{U}_w$.
To see the claim, note that $P_{B_1} + P_{B_2} + ... + P_{B_k} = I$ which implies $s_w(P_{B_1}) + s_w(P_{B_2}) + ... + s_w(P_{B_k}) = 1$. But s_w is a homomorphism $\Rightarrow$ $s_w(P_{w_l}) \in \{0, 1\}$. Therefore, $\exists$ i such that $s_w(P_{w_i}) = 1$ which further implies that P_{B_i} corresponds to a function $\chi_{\bar{B_i}} = \chi_{\bar{U_i}}$ where $B_i = U_i \cap \mathbb{N}, w \in U_i$. This establishes the claim.
Thus we have that $s_w(P_{B_i}) = 1 \Rightarrow (t_i - \epsilon) = s_w((t_i-\epsilon)P_{B_i}) \leq s(P_{B_i}HP_{B_i}) \leq s_w((t_i+\epsilon)P_{B_i}) \leq t_i + \epsilon$ where s is any state on $B(l^2(\mathbb{N}))$ that extends s_w.
$\Rightarrow t_i - \epsilon \leq l_{s_w}(H) \leq u_{s_w}(H) \leq t_i + \epsilon$
$\Rightarrow u_{s_w}(H) - l_{s_w}(H) \leq 2\epsilon \;\forall\; \epsilon$
$\Rightarrow u_{s_w}(H) = l_{s_w}(H) \;\forall\; w$.
This proves the theorem. □

Anderson saw we could make k independent of H !.
Let $\mathbb{N} \times \mathbb{N} = \bigcup_{i \in \mathbb{N}} \mathbb{N}_i$ where $N_i = \{(i,j) \in \mathbb{N} \times \mathbb{N} : j \in \mathbb{N}\}$.

Lemma 7.19. *Fix $\psi : \bigcup_{i\in\mathbb{N}} \mathbb{N}_i \to \mathbb{N}$ 1-1, onto map. Let $H_i \in \mathcal{H}[a,b]$ and let $H = U_\psi^*(H_1 \oplus H_2 \oplus ... \oplus ...)U_\psi$, where $U_\psi : l^2(\mathbb{N}) \to \sum_i \oplus l^2(N_i)$ is a unitary. Then for each $\epsilon > 0$, $\exists$ finite collection of disjoint sets, $B_1, B_2, ..., B_k$ with $B_1 \cup B_2 \cup ... \cup B_k = \mathbb{N}$ and $t_1, t_2, ..., t_k \in \mathbb{R}$ such that $(t_i - \epsilon)P_{B_l} \le P_{B_l} H P_{B_l} \le (t_i + \epsilon)P_{B_l}$ for every $l \Leftrightarrow \ \forall i \ \exists \ t_i, t_2, ..., t_k$ and $B_1^i \cup B_2^i \cup ... \cup B_l^i = N_i$ such that $(t_l - \epsilon)P_{B_l^i} \le P_{B_l^i} H_i P_{B_l^i} \le (t_i + \epsilon)P_{B_l^i}$.*

Definition 7.20. *We will call a collection of sets $B_1, \dots, B_k$ and real numbers $t_1, \dots, t_k$ that satisfy the conclusion of* **??** *a generalized (ϵ, k)-paving of H.*

The space $\mathcal{U} = \bigcap_{\omega\in\beta\mathbb{N}} \mathcal{U}(\omega)$ will be called the uniqueness domain. Note that $\mathcal{U}$ is an operator system and that an affirmative answer to the Kadison-Singer problem is equivalent to the condition that $\mathcal{U} = B(\ell^2(\mathbb{N}))$.

Theorem 7.21. *The following are equivalent.*

(1) $\mathcal{U} = B(\ell^2(\mathbb{N}))$
(2) *for all $\epsilon > 0$, there exists k such that every $H \in \mathcal{H}[a,b]$ has a generalized (ϵ, k)-paving.*
(3) *for all $\epsilon > 0$ and for all $H \in \mathcal{H}[a,b]$ there exists k such that H has a generalized (ϵ, k)-paving.*

Proof. 1) implies 3). If $\mathcal{U} = B(\ell^2(\mathbb{N})$, then every state on the diagonal extends uniquely to H and so by the last theorem 3) follows.

3) implies 1) By the last theorem every state extends uniquely to every self-adjoint element $H \in \mathcal{H}[a,b]$. Given $H = H^*$ there exists α, β such that $\alpha H + \beta I \in \mathcal{H}[a,b]$ and so $\alpha H + \beta I \in \mathcal{U}$. Hence,

$$H = \alpha^{-1}(\alpha H + \beta I - \beta I) \in \mathcal{U}.$$

Since $\mathcal{U}$ is an operator system that contains every self-adjoint element of $B(\ell^2(\mathbb{N}))$, we see that $\mathcal{U} = B(\ell^2(\mathbb{N}))$.

2) implies 3). This is clear.

3) implies 2). Suppose that 3) is true but 2) is false. There exists $\epsilon > 0$ and a sequence $H_n = H_n^* \in \mathcal{H}[a,b]$ such that H_n can be (ϵ, k_n)-paved but not (ϵ, k)-paved for $k < k_n$ and $k_n \to \infty$ as $n \to \infty$.

Let $H = U_\phi^*(H_1 \oplus \dots)U_\phi$, U_ϕ is the unitary in [REF]. $H \in \mathcal{H}[a,b]$ and hence by 3) there exists a generalized (ϵ, k_∞)-paving of H. By the lemma each of the operators in the direct sum has a generalized (ϵ, k_∞)-paving. This contradiction proves our result. ☐

We now descibe an expectation operator from $B(\ell^2(\mathbb{N}))$ onto $\mathcal{D}$. Given an arbitrary bounded operator $T = (t_{i,j}) \in B(\ell^2(\mathbb{N}))$, define $E(T) = \text{diag}(t_{i,i}) \in \mathcal{D}$. The operator E is linear, $E(D) = D$ for all $D \in \mathcal{D}$, if $T \ge 0$, then $E(T) \ge 0$, $\|E(T)\| \le \|T\|$ and $E \circ E = E$. The porjection E is called the a conditional expectation of $B(\ell^2(\mathbb{N}))$ onto the diagonal $\mathcal{D}$.

Proposition 7.22. *Given $s_\omega : \mathcal{D} \to \mathbb{C}$ a pure state, $s_\omega \circ E$ is a state that extends s_ω.*

Proof. $s_\omega(E(I)) = s_\omega(I) = 1$. If $T \geq 0$, then $E(T) \geq 0$ and so $s_\omega \circ E(T) \geq q0$. □

This is called the canonical extension of s_ω to $B(\ell^2(\mathbb{N}))$. The Kadison-Singer problem is equivalent to asking if $s_\omega \circ E$ is the only extension of s_ω. Note that $T - E(T)$ is a bounded linear operator with 0 diagonal and if we use the canonical extension we see that

$$s_\omega \circ E(T - E(T)) = s_\omega \circ E(T) - s_\omega \circ E(E(T)) = 0.$$

Conversely, suppose that s is any extension of s_ω such that $s(X) = 0$ for all X such that $E(X) = 0$. For any T, $s(T - E(T)) = 0$ and so $s(T) = S(E(T)) = s_\omega(T) = s_\omega \circ E(T)$. We have proven the following simple result.

Proposition 7.23. *The Kadison-Singer problem has an affirmative answer if and only if every state that extends s_ω is 0 on the $B(\ell^2(\mathbb{N}))_0 = \{X \in B(\ell^2(\mathbb{N})) : E(X) = 0\}$.*

The proof of the following result follows from the results that we have proved up to this point.

Proposition 7.24. *Fix a pure state s_ω on the diagonal. The following are equivalent.*

(1) *s_ω has a unique extension*
(2) *$s(H) = 0$ for all $H = H^* \in B(\ell^2(\mathbb{N}))_0$*
(3) *$l_{s_\omega}(H) = u_{s_\omega}(H) = 0$ for all $H = H^* \in B(\ell^2(\mathbb{N}))_0$*
(4) *For all $\epsilon > 0$, there exists $A \in \mathcal{U}_\omega$ such that $-\epsilon P_A \leq P_A H P_A \leq \epsilon P_A$.*

Note that if $H = H^*$, then $-\epsilon P_A \leq P_A H P_A \leq \epsilon P_A$ is equivalent to $\|P_A H P_A\| \leq \epsilon$.

Definition 7.25. *Given $H = H^* \in B(\ell^2(\mathbb{N}))$ we say that H has an (ϵ, k)-paving if there exists disjoint sets $B_1, \ldots, B_k \subseteq \mathbb{N}$ such that $B_1 \cup \ldots B_k = \mathbb{N}$ and $\|P_{B_i} H P_{B_i}\| \leq \epsilon$. Let $\mathcal{H}_0[-1,1] = \mathcal{H}[-1,1] \cap B(\ell^2(\mathbb{N}))_0$.*

Theorem 7.26 (Anderson)**.** *The following are equivalent*

(1) *The Kadison-Singer problem is true, i.e. $\mathcal{U} = B(\ell^2(\mathbb{N}))$.*
(2) *For all $\epsilon > 0$, there exists $k \in \mathbb{N}$ such that every $H \in \mathcal{H}_0[-1,1]$ can be (ϵ, k)-paved.*
(3) *For all $\epsilon > 0$ every $H \in \mathcal{H}_0[-1,1]$ can be (ϵ, k)-paved.*
(4) *There exists $r < 1$ and $k \in \mathbb{N}$ such that every $H \in \mathcal{H}_0[-1,1]$ can be (r, k)-paved*
(5) *There exists $r < 1$ such that every $H \in \mathcal{H}_0[-1,1]$ can be (r, k)-paved*
(6) *for each $H \in \mathcal{H}_0[-1,1]$ there exists $r < 1$ and $k \in \mathbb{N}$ such that H can be (r, k)-paved.*

Proof. The equivalence of (2) and (3) and the equivalence of (4)and (5) follows from the direct sum lemma as in earlier proofs. Clearly, (5) implies (6).

(3)implies (1). Let $\omega \in \beta\mathbb{N}$ and s be a state extension of s_ω. Given $H \in \mathcal{H}_0[-1,1]$ and $\epsilon > 0$, the there exists a finite collection of disjoint sets such that $A_1 \cup A_2 \cup \ldots \cup A_K = \mathbb{N}$ and $\|P_{A_i} H P_{A_i}\| < \epsilon$. This implies that $-\epsilon P_{A_i} \le P_{A_i} H P_{A_i} \le \epsilon P_{A_i}$ and thus $-\epsilon s_\omega(P_{A_i}) \le s_\omega(P_{A_i} H P_{A_i}) \le \epsilon s_\omega(P_{A_i})$ We know that there exists an i_0 such that $P_{A_{i_0}} \in \mathcal{U}_\omega$ and $s_\omega(P_{A_{i_0}}) = 1$. So we have $|s_\omega(P_{A_{i_0}} H P_{A_{i_0}})| \le \epsilon$ which implies that $s_\omega(P_{A_{i_0}} H P_{A_{i_0}}) = s_\omega(H)$. Therefore $|s(H)| \le \epsilon$ for all $\epsilon > 0$. This implies that $s(H) = 0$ for all $H \in \mathcal{H}_0[-1,1]$ and for all s extending s_ω. Given any H, $H - E(H) \in \mathcal{H}_0[-r,r]$ which implies that $\frac{1}{r}(H - E(H)) \in \mathcal{H}_0[-1,1]$. Thus, $s(\frac{1}{r}(H - E(H)) = 0$ which implies $s(H - E(H)) = 0$ or $s(H) = s(E(H))$.

(1)implies (3). We have seen that (1) implies every $H \in \mathcal{U}_\omega$. When $H \in \mathcal{H}_0[-1,1]$, by proposition 7.24, there exists $A \in \mathcal{U}_\omega$ such that $\|P_A H P_A\| \le \epsilon$. As before for each $\omega \in \beta\mathbb{N}$, we get an $A_\omega \in \mathcal{U}_\omega$ such that $\|P_{A_\omega} H P_{A_\omega}\| \le \epsilon$. Each $A_\omega = U_\omega \cap \mathbb{N}$, U_ω is an open neighborhood of ω. Choose finitely many that cover and then make disjoint.

(2) implies (4) is trivial.

(4)implies (2). Given $\epsilon > 0$, choose l such that $r^l < \epsilon$. Given H, there exists finitely many disjoint sets such that $A_1 \cup A_2 \cup \ldots \cup A_K = \mathbb{N}$ and $\|P_{A_j} H P_{A_j}\| \le r$. Thus, $\frac{1}{r} P_{A_j} H P_{A_j} \in \mathcal{H}_0[-1,1]$. So there exists finitely many disjoint sets such that $B_1^j \cup B_2^j \cup \ldots \cup B_k^j = \mathbb{N}$ and $\|\frac{1}{r} P_{B_i^j}(P_{A_j} H P_{A_j}) P_{B_i^j}\| \le r$. This implies that $\|P_{B_i^j \cap A_j} H P_{B_i^j \cap A_j}\| \le r^2$. We now have k^2 disjoint subsets, $C_1 \cup C_2 \cup \ldots \cup C_{k^2} = \mathbb{N}$ such that $\|P_{C_i} H P_{C_i}\| \le r^2$. Inductively obtain k^l disjoint subsets such that $E_1 \cup E_2 \cup \ldots \cup E_{k^l} = \mathbb{N}$ and $\|P_{E_i} H P_{E_i}\| \le r^l$. Therefore every $H \in \mathcal{H}_0[-1,1]$ can be (ϵ, k^l) paved, k^l depends only on ϵ.

Thus, statements (1)–(5) are equivalent and imply (6). We now show that (6) implies (5). Suppose that (6) is true but that (5) is not. Then statement (5) fails to hold for $r_n = 1 - 1/n$. Hence, there exists $H_n \in \mathcal{H}_0[-1,1]$, which can not be (r_n, k)-paved for any k. Again we use the direct sum lemma, let $H = U_\phi^*(H_2 \oplus H_3 \oplus \ldots)U_\phi \in \mathcal{H}_0[-1,1]$. By (6) there exists some $r < 1$, and $k \in \mathbb{N}$ such that H can be (r,k)-paved, and hence, each H_n can be (r,k)-paved. But for $r < r_n$ this is a contradiction. Hence, (6) implies (5). □

Given $H \in \mathcal{H}_0[-1,1]$ and k, let $p_k(H) = inf\{\max_{1 \le l \le k} \|P_{A_l} H P_{A_l}\| : A_1 \cup A_2 \cup \ldots \cup A_K = \mathbb{N}, A_i \cap A_j = \phi, i \ne j\}$. If Kadison Singer Problem is true, given $\epsilon > 0$, there exists a k such that $p_k(H) < \epsilon$ for all H. This implies that $\sup_{H \in \mathcal{H}_0[-1,1]} p_k(H) \le \epsilon$.

Some important things that we know are:

1. $\sup_{H \in \mathcal{H}_0[-1,1]} p_2(H) = 1$ that is 2-paving fails.
2. Do not know if the $p_3 = \sup_{H \in \mathcal{H}_0[-1,1]} p_3(H) < 1$ or not. Weiss - Zarikian shown $p_3 \ge 0.92$

Definition 7.27. *Let* $\mathcal{R}_0 = \{H = H^* : H^2 = I, E(H) = 0\}$

Note: $H^2 = I$, $\sigma(H) \subseteq \{-1, 1\}$. This implies that $\mathcal{R}_0 \subseteq \mathcal{H}_0[-1, 1]$. Geometrically these are reflections as there exists $l_+, l_- \in l^2(\mathbb{N})$ such that $l_+ \perp l_-$ and $l_+ \oplus l_- = l^2(\mathbb{N})$. Thus, $H(h_+ + h_-) = h_+ - h_-$, reflection about l_+.

Theorem 7.28. *The following are equivalent*

(1) *Kadison Singer Problem is true*
(2) *For all $\epsilon > 0$, there exists k such that every $H \in \mathcal{R}_0$ can be (ϵ, k) paved.*
(3) *For all $\epsilon > 0$, every $H \in \mathcal{R}_0$ can be (ϵ, k) paved.*
(4) *There exists $r < 1$ and $k \in \mathbb{N}$ such that every $H \in \mathcal{R}_0$ can be (r, k)-paved*
(5) *There exists $r < 1$ such that every $H \in \mathcal{R}_0$ can be (r, k)-paved*

Definition 7.29. *Let* $\mathcal{P}_{\frac{1}{2}} = \{P \in \mathcal{B}(l^2(\mathbb{N})) : P = P^* = P^2, E(P) = \frac{1}{2}I\}$

Theorem 7.30 (CEKPT)**.** *The following are equivalent*

(1) *The kadison-Singer problem is true.*
(2) *For all ϵ, there exists k such that for every P in $\mathcal{P}_{\frac{1}{2}}$, there exists disjoint sets $C_1, C_2, \dots, C_k$ with $C_1 \cup \cdots \cup C_k = \mathbb{N}$ such that for each l, $(\frac{1}{2} - \epsilon)P_{C_l} \le P_{C_l} P P_{C_l} \le (\frac{1}{2} - \epsilon)P_{C_l}$.*
(3) *There exists $r < \frac{1}{2}$ and k such that for every P in $\mathcal{P}_{\frac{1}{2}}$, there exists disjoint sets $C_1, C_2, \dots, C_k$ with $C_1 \cup \cdots \cup C_k = \mathbb{N}$ such that for each l, $(\frac{1}{2} - r)P_{C_l} \le P_{C_l} P P_{C_l} \le (\frac{1}{2} - r)P_{C_l}$.*
(4) *There exists $r < \frac{1}{2}$ such that each $P \in \mathcal{P}_{\frac{1}{2}}$, there exists k ,disjoints sets $C_1, \dots, C_k$ with $C_1 \cup \cdots \cup C_k = \mathbb{N}$ such that $(\frac{1}{2} - r)P_{C_l} \le P_{C_l} P P_{C_l} \le (\frac{1}{2} - r)P_{C_l}$.*
(5) *for each $P \in \mathcal{P}_{\frac{1}{2}}$ there exists $r < 1/2$ and $k \in \mathbb{N}$, such that there exists k disjoint subsets, $C_1 \cup \cdots C_k = \mathbb{N}$ with $(\frac{1}{2} - r)P_{C_l} \le P_{C_l} P P_{C_l} \le (\frac{1}{2} + r)P_{C_l}$, for $1 \le l \le k$.*

Proof. $(1 \Rightarrow 2)$ Let $P \in \mathcal{P}_{\frac{1}{2}}$. Then $2P - I = U \in \mathcal{R}_0$. There by the last theorem given an $\epsilon > 0$ there exists k and disjoints sets $C_1, \dots, C_k$ with $C_1 \cup \cdots \cup C_k = \mathbb{N}$ such that $-2\epsilon P_{C_l} \le P_{C_l} U P_{C_l} \le 2\epsilon P_{C_l}$.
$\implies -2\epsilon P_{C_l} \le P_{C_l}(2P - I)P_{C_l} \le 2\epsilon P_{C_l}$
$\implies (1 - 2\epsilon)P_{C_l} \le P_{C_l}(2P)P_{C_l} \le (1 + 2\epsilon)P_{C_l}$
$\implies \frac{1-2\epsilon}{2} P_{C_l} \le P_{C_l} P P_{C_l} \le \frac{1+2\epsilon}{2} P_{C_l}$
$\implies (\frac{1}{2} - \epsilon)P_{C_l} \le P_{C_l} P P_{C_l} \le (\frac{1}{2} + \epsilon)P_{C_l}$.
$(2 \Rightarrow 3)$ and $(3 \Rightarrow 4)$ are clearly true.
$(4 \Rightarrow 1)$ Given $U \in \mathcal{R}_0$, let $P = \frac{U+I}{2} \in \mathcal{P}_{\frac{1}{2}}$. then there exists k and disjoint sets $C_1, \dots, C_k$ with $C_1 \cup \cdots \cup C_k = \mathbb{N}$ such that $(\frac{1}{2} - r)P_{C_l} \le P_{C_l}(\frac{U+I}{2})P_{C_l} \le (\frac{1}{2} + r)P_{C_l}$

$\implies -rP_{C_l} \leq P_{C_l}(\frac{U}{2})P_{C_l} \leq rP_{C_l}$
$\implies \|P_{C_l}UP_{C_l}\| \leq 2r < 1.$
Hence, this 4 implies 4 from last theorem which further implies the Kadison-Singer problem.

Thus, we have shown that (1)–(4) are equivalent. Clearly, (4) implies (5) and we complete the proof by showing that (5) implies (4).

To this end suppose that (5) is true, but that (4) is false. Then for each $r_n = 1/2 - 1/n, n \geq 3$, there exists $P_n \in \mathcal{P}_{\frac{1}{2}}$ which can not be r_n-paved in the above sense for any $k \in \mathbb{N}$. Form the operator $P = U_\phi^*(P_3 \oplus P_4 \oplus \ldots)U_\phi$. Note that $P \in \mathcal{P}_{\frac{1}{2}}$ and so this operator has a paving of the above type for some $r < 1/2$ and some $k \in \mathbb{N}$, and hence each P_n will have an (r, k)-paving of the desired type. But as soon as $r < r_n$ this is a contradiction, and, hence, (5) implies (4).

□

8. Introduction to frames in Hilbert spaces

Definition 8.1. *Let $\mathcal{H}$ be a Hilbert space. A set $\{f_i\}_{i\in I}$ in $\mathcal{H}$ is called a **Riesz basis** for $\mathcal{H}$ if there exists $\{u_i\}_{i\in I}$, an orthonormal basis for $\mathcal{H}$ and an invertible, bounded linear operator $S \in \mathcal{B}(\mathcal{H})$ such that for each i, $f_i = S(u_i)$*

Definition 8.2. *A set $\{f_i\}_{i\in I}$ in $\mathcal{H}$ is called a **Riesz basis set** if it is a Riesz basis for $\mathcal{H}_0 = span\{f_i : i \in I\}$ and a* **Riesz basic sequence** *when the index set is countable.*

Definition 8.3. *A set $\{f_i : i \in I\}$ in $\mathcal{H}$ is called a **Bessel set** if there exists a constant B such that*

$$\textstyle\sum_{i\in I} |\langle x, f_i\rangle|^2 \leq B\|x\|^2, \text{ for each } x \text{ in } \mathcal{H}.$$

If the index set is countable, then we call a Bessel set a **Bessel sequence.**

Note that saying $\{f_i\}_{i\in I}$ is a Bessel set is equivalent to saying that the map $F : \mathcal{H} \longrightarrow l^2(I)$ given by $F(x) = (\langle x, f_i\rangle)_{i\in I}$ is bounded with $\|F\|^2 \leq B$. In this case,

$$\langle F^*(e_i), x\rangle = \langle e_i, (\langle x, f_i\rangle)_{i\in I}\rangle = \overline{\langle x, f_i\rangle} = \langle f_i, x\rangle$$

Therefore $F^*(e_i) = f_i$ for each i $\in I$.
So, Bessel is same as $e_i \longrightarrow f_i$ extends to a bounded map from $l^2(I)$ to $\mathcal{H}$. F^* is called **synthesis operator**.

Definition 8.4. *A set $\{f_i\}_{i\in I}$ in $\mathcal{H}$ is called a **frame** if there exists A, B with $0 < A \leq B$ such that*

$$A\|x\|^2 \leq \textstyle\sum_{i\in I} |\langle x, f_i\rangle|^2 \leq B\|x\|^2 \text{ for each } x \in \mathcal{H}$$

Equivalently, $\{f_i\}_{i\in I}$ is a frame if and only if $\{f_i\}_{i\in I}$ is Bessel and F is bounded below. When the index set for a frame is bounded below, we will refer to the set as a **frame sequence.**

Example 6 Let $\{u_i\}_{i\in I}, \{v_j\}_{j\in J}$ be orthonormal basis for a Hilbert space $\mathcal{H}$. Then for each $x \in \mathcal{H}$ we have

$$\sum_{i,j}(|\langle x, u_i\rangle|^2 + |\langle x, v_j\rangle|^2) = 2\|x\|^2$$

Therefore $\{u_i\}_{i\in I} \cup \{v_j\}_{j\in J}$ is a frame for $\mathcal{H}$ with $A = 2, B = 2$
Example 7 Let $\{f_i\}_{i\in I}$ be a frame.$\{f_i\}_{i\in I} \cup \{0\}$ is still a frame for $\mathcal{H}$.

Definition 8.5. *A frame is called a* ***tight frame*** *if $A = B$ and is called a* ***Parseval frame*** *if $A = B = 1$.*

Equivalently,

$$\{f_i\}_{i\in I} \text{ is a Parseval frame.} \Leftrightarrow \text{F is an isometery.} \Leftrightarrow F^*F = I.$$
$$\Leftrightarrow F^*F = P_{\mathcal{R}(F)}.$$

Note: $\{f_i\}_{i\in I}$ is a frame iff Bessel and $F^*F \geq AI$.

Proposition 8.6. *Let $\{f_i\}_{i\in I}$ is a Riesz basis for a Hilbert space $\mathcal{H}$. Then $\{f_i\}_{i\in I}$ is a frame for $\mathcal{H}$.*

Proof. Let $S \in \mathcal{B}(\mathcal{H})$ be an invetible operator and $\{u_i\}_{i\in I}$ be an orthonormal basis for $\mathcal{H}$ so that for each i, $S(u_i) = f_i$. Then

$$\sum_{i\in I} |\langle x, f_i\rangle|^2 = \sum_{i\in I} |\langle x, S(u_i)\rangle|^2 = \sum_{i\in I} |\langle S^*(x), u_i\rangle|^2 = \|S^*(x)\|^2$$

As, $S \in \mathcal{B}(\mathcal{H})$ and is invertible therefore

$$\|S^*(x)\|^2 \leq \|S^*\|^2\|x\|^2, \|S^*(x)\| \geq \frac{\|x\|}{\|S^{*-1}\|}.$$

Hence $\{f_i\}_{i\in I}$ is a frame with $A = \|S^{*-1}\|^{-2}, b = \|S^*\|^2$.

□

Proposition 8.7. *Let $\{f_j\}_{j\in J} \subseteq \mathcal{H}$, then $\{f_j\}_{j\in J}$ is a Riesz basis for $\mathcal{H}$ if and only if $\{f_j\}_{j\in J}$ is a Bessel set whose closed linear span is $\mathcal{H}$ and there exists $c > 0$ such that $FF^* \geq cI_{\ell^2(J)}$.*

Proof. First, we assume that the set is a Riesz basis, so that there exists an orthonormal basis for $\mathcal{H}, \{u_j\}_{j\in J}$ and a bounded invertible operator $S : \mathcal{H} \to \mathcal{H}$ such that for all $j, f_j = Su_j$. Clearly, the closed span of $\{f_j\}$ is all of $\mathcal{H}$ and by the above result or by noting that $\sum_j |\langle x, f_j\rangle|^2 = \sum |\langle S^*(x), f_j\rangle|^2 = \|S^*(x)\|^2 \leq \|S\|^2\|x\|^2$, we see that $\{f_j\}$ is a Bessel set.

Let $U : \ell^2(J) \to H$ be the unitary operator uniquely defined by $U(e_j) = u_j$. Since $F^*(e_j) = f_j = S(u_j) = U(e_j)$, we have that $F^* = SU$, and so $FF^* = U^*S^*SU$. Now, $(S^{-1})^*S^{-1} \leq \|S^{-1}\|^2 I_{\mathcal{H}}$, and hence, $I_{\mathcal{H}} = S^*(S^*)^{-1}S^{-1}S \leq \|S^{-1}\|^2 S^*S$. Finally, $\|S^{-1}\|^{-2}I_{\ell^2(J)} = \|S^{-1}\|^{-2}U^*I_{\mathcal{H}}U \leq U^*S^*SU = F^*F$, and we have the desired lower bound.

Conversely, assume that $\{f_j\}$ satisfies the three conditions. Since it is a Bessel set, F and F^* are bounded and the condition, $cI_{\mathcal{H}} \leq FF^*$ implies that F^* is bounded below and so its range, $\mathcal{R}(F^*)$, is closed. But since each $f_j = F^*(e_j)$ is in the range we have that $\mathcal{R}(F^*) = \mathcal{H}$. Hence, F^* is one-to-one and onto. Let $U = F^*(F^*F)^{-1/2}$, then $U^*U = I_{\ell^2(J)}$, so that U is an isometry, but U is also invertible, and hence, $U : \ell^2(J) \to \mathcal{H}$ is a unitary. Let $u_j = U(e_j), j \in J$, since U is unitary this set is an orthonormal basis for $\mathcal{H}$. Finally, $S = F^*U^* : \mathcal{H} \to \mathcal{H}$ is invertible and $S(u_j) = F^*U^*Ue_j = f_j$. □

Corollary 8.8. *Let $\{f_j\}_{j\in J} \subseteq \mathcal{H}$. Then $\{f_j\}_{j\in J}$ is a Riesz basic set if and only if $\{f_j\}_{j\in J}$ is a Bessel set and there exists $c > 0$ such that $FF^* \geq cI_{\ell^2(J)}$.*

Proof. Let $\mathcal{H}_0$ be the closed linear span of $\{f_j\}_{j\in J}$ and let F_0 denote the restriction of F to $\mathcal{H}_0$, then $F_0^*(e_j) = f_j$. Note that for $x \perp \mathcal{H}_0, F(x) = 0$.

Now $\{f_j\}_{j\in J}$ is a Riesz basic set if and only if $\{f_j\}_{j\in J}$ is a Riesz basis for $\mathcal{H}_0$, which by the above result is equivalent to $\{f_j\}_{j\in J}$ being a Bessel set and $F_0F_0^* \geq cI_{\ell^2(J)}$, for some $c > 0$. Finally, note that $FF^*(e_j) = F(f_j) = F_0(f_j) = F_0F_0^*(e_j)$, and so $FF^* = F_0F_0^*$, and the result follows. □

A Bessel set $\{f_j\}_{j\in J} \subseteq \mathcal{H}$ is called **bounded(below)** if there exists a constant, $\delta > 0$, such that for all $j, \delta \leq \|f_j\|$. Note that if $\sum_j |\langle x, f_j\rangle|^2 \leq B\|x\|^2$, for every $x \in \mathcal{H}$, then $\delta^2\|f_i\|^2 \leq \|f_i\|^4 \leq \sum_j |\langle f_i, f_j\rangle|^2 \leq B\|f_i\|^2$, and hence $\delta \leq \sqrt{B}$. We shall call a bounded Bessel sequence $\{f_j\}_{j\in bbN}$ with upper Bessel bound B and lower bound δ a (B, δ)-Bessel sequence.

When we say that a sequence $\{f_j\}_{j\in\mathbb{N}}$ can be **partitioned into k Riesz basic sequences** we mean that there exists a partition of $\mathbb{N}$ into k subsets, $C_1 \cup C_2 \cup \cdots C_k = \mathbb{N}$, such that $\{f_j\}_{j\in C_l}$ is a Riesz basic sequence for $1 \leq l \leq k$.

The equivalence of (1), (3) and (4) were proven by Casazza and Tremain in [**?**] and by Casazza, Fickus, Tremain and Weber[**?**].

Theorem 8.9 (Casazza and Tremain)**.** *The following are equivalent:*

1. *Kadison-Singer is true,*
2. *for each $0 < \delta \leq \sqrt{B}$, there exists $k \in \mathbb{N}$, such that every (B, δ)-bounded Bessel sequence can be partitioned into k Riesz basic sequences,*
3. *each bounded Bessel sequence can be partitioned into finitely many Riesz basic sequences,*
4. *each bounded frame sequence can be partitioned into finitely many riesz basic sequences.*

Proof. It is clear that (2) implies (3) and that (3) implies (4). We begin by proving that (1) implies (2).

Fix $0 < \epsilon < \delta^2/4$ and since Kadison-Singer is true by there is a k so that every $H \in \mathcal{H}[0, B]$ has a generalized (ϵ, k)-paving. If $\{f_j\}_{j\in\mathbb{N}}$ is a (B, δ)-bounded Bessel sequence, then $FF^* = (\langle f_j, f_i\rangle)$ satisfies $0 \leq FF^* \leq BI_{\ell^2(\mathbb{N})}$. Hence, there exists a partition $C_1 \cup C_2 \cup \cdots C_k = \mathbb{N}$, and real numbers, $t_1, t_2, ..., t_k$ such that $(t_l - \epsilon)P_{C_l} \leq P_{C_l}FF^*P_{C_l} \leq (t_l + \epsilon)P_{C_l}$. Taking $j \in C_l$ and considering the corresponding (j, j)-entry, yields $t_l - \epsilon \leq \|f_j\|^2 \leq t_l + \epsilon$. Hence, $\delta^2 \leq \|f_j\|^2 \leq t_l + \epsilon$. Thus, $0 < \delta^2/2 \leq \delta^2 - 2\epsilon \leq t_l - \epsilon$, and so $\frac{\delta^2}{2}P_{C_l} \leq P_{C_l}FF^*P_{C_l}$.

Hence, we have that each of the Bessel sequences, $\{f_j\}_{j\in C_l}$ has analysis operator F_l satisfying $F_lF_l^* = (\langle f_j, f_i\rangle)_{i,j\in C_l} \geq \frac{\delta^2}{2} I_{\ell^2(C_l)}$, and thus, is a Riesz basic sequence.

We now prove that (4) implies (1), by showing that (4) implies condition (5) in Theorem 7.30. To this end, let $P \in \mathcal{P}_{1/2}$, so that $P = P^* = PP^*$. If we let $\{f_j\}_{j\in\mathbb{N}}$ denote the columns of P(which are also the rows of P), then $P = (\langle f_j, f_i\rangle)$. Let $\mathcal{H} = \mathcal{R}(P) \subseteq \ell^2(\mathbb{N})$. Note that $\{f_j\}$ is a Parseval frame sequence for $\mathcal{H}$, since for $h \in \mathcal{H}$, we have that $h = P(h) = (\langle h, f_j\rangle)$. Also, since $1/2 = p_{j,j} = \|f_j\|^2$, we have that $\{f_j\}$ is a bounded Parseval frame.

Hence, by the hypothesis, there exists a finite partition of $\mathbb{N} = C_1 \cup C_2 \cup \cdots \cup C_n$, such that each set $\{f_j\}_{j\in C_l}$ is a Riesz basic sequence. This implies that there exists $c_l > 0$, such that $(\langle f_j, f_i\rangle)_{i,j\in C_l} \geq c_l I_{\ell^2(C_l)}$, which is equivalent to $P_{C_l}PP_{C_l} \geq c_lP_{C_l}$. Note that we may assume that $c_l < 1/2$.

Applying similar reasoning to $Q = I - P \in \mathcal{P}_{1/2}$, we get a partition, $\mathbb{N} = D_1\cup D_2\cup\cdots D_m$, and numbers $0 < d_k < 1/2$, such that $P_{D_k}(I-P)P_{D_k} \geq d_kP_{D_k}$, which is equivalent to $(1-d_k)P_{D_k} \geq P_{D_k}PP_{D_k}$.

Thus, we have that $(1-d_k)P_{C_l\cap D_k} = P_{C_l}(1-d_k)P_{D_k}P_{C_l} \geq P_{C_l}P_{D_k}PP_{D_k}P_{C_l} = P_{C_l\cap D_k}PP_{C_l\cap D_k}$, and $P_{C_l\cap D_k}PP_{C_l\cap D_k} = P_{D_k}P_{C_l}PP_{C_l}P_{D_k} \geq P_{D_k}c_lP_{C_l}P_{D_k} = c_lP_{C_l\cap D_k}$.

Letting $r = min\{c_1, ..., c_m, d_1, ..., d_n\}$, and $E_{l,k} = C_l\cap D_k$, which is a finite partition of $\mathbb{N}$ and setting $\epsilon = 1/2-r$, we have that $(1/2-\epsilon)P_{E_{l,k}} = rP_{E_{l,k}} \leq P_{E_{l,k}}PP_{E_{l,k}} \leq (1-r)P_{E_{l,k}} = (1/2-\epsilon)P_{E_{l,k}}$ and so condition (5) is met. □

There are many more equivalences worked out in [**?**]. In particular, it is shown that Kadison-Singer is true if and only if every bounded Bessel sequence is a finite union of frame sequences.

9. Further Paving Results

In this section we discuss some other paving results that are related to the Kadison-Singer conjecture. The first results show that it is enough to consider paving of strictly upper triangular operators and use some new ideas that came from function theory and they are taken from the paper [**?**].

Lemma 9.1. *Let $\mathcal{B}$ be a unital C^*-algebra and let $s : \mathcal{B} \to \mathbb{C}$ be a state. If $p \in \mathcal{B}$ is positive and invertible, then $s(p)s(p^{-1}) \geq 1$.*

Proof. Let q be positive and invertible, and let $t \in \mathbb{R}$, then $0 \leq s((tq + q^{-1})^2) = t^2s(q^2) + 2t + s(q^{-2})$. Hence, this second degree polynomial either has no real roots or a repeated real root and so $4 - 4s(q^2)s(q^{-2}) \leq 0$. Thus, $1 \leq s(q^2)s(q^{-2})$, and the result follows by letting $q = \sqrt{p}$. □

Theorem 9.2 (P-Raghupathi)**.** *Fix $0 < a < 1 < b$, let $\mathcal{B}$ be a unital C^*-algebra and let $s_i : \mathcal{B} \to \mathbb{C}, i = 1, 2$ be states. Then the following are equivalent:*

(1) $s_1 = s_2$,

(2) *for every positive invertible* $p \in \mathcal{B}, s_1(p)s_2(p^{-1}) \geq 1$,
(3) *for every* $p \in \mathcal{H}[a,b], s_1(p)s_2(p^{-1}) \geq 1$.

Proof. If $s_1 = s_2$, then (2) holds by the above lemma. Clearly, (2) implies (3). It remains to show that (3) implies (1).

To this end let $h = h^* \in \mathcal{B}$, so that for t in some neighborhood of 0, we will have that $e^{th} \in \mathcal{H}[a,b]$. This implies that the real analytic function, $f(t) = s_1(e^{th})s_2(e^{-th}) \geq 1$. Since $f(0) = 1$, we see that $t = 0$ is a critical point of the function. Hence, $0 = f'(0) = s_1(h) - s_2(h)$. Thsu, $s_1(h) = s_2(h)$, for every $h = h^*$, and it follows that $s_1 = s_2$. □

Given an operator system $\mathcal{S} \subseteq \mathcal{B}$, and a state $s : \mathcal{S} \to \mathbb{C}$, recall the definition of $l_s(h)$ and of $u_s(h)$.

Theorem 9.3 (P-Raghupathi)**.** *Let* $\mathcal{B}$ *be a unital C*-algebra, let* $\mathcal{S} \subseteq \mathcal{B}$ *be an operator system, let* $s : \mathcal{S} \to \mathbb{C}$ *be a state and fix 0 ¡ a ¡ 1 ¡ b. Then the following are equivalent:*

(1) s *extends uniquely to a state on* $\mathcal{B}$,
(2) *for every positive, invertible* $p \in \mathcal{B}, l_s(p)l_s(p^{-1}) \geq 1$,
(3) *for every* $p \in \mathcal{H}[a,b], l_s(p)l_s(p^{-1}) \geq 1$.

Proof. Assuming (1), let $\hat{s} : \mathcal{B} \to \mathbb{C}$ be the unique extension of s. Then $\hat{s}(p) = l_s(p)$ and $\hat{s}(p^{-1}) = l_s(p^{-1})$, and (2) follows.

Clearly, (2) implies (3), so it remains to show that (3) implies (1). Let $s_i : \mathcal{B} \to \mathbb{C}, i = 1,2$ be states that extend s. Then for any $p \in \mathcal{H}[a,b]$, we have that $s_1(p)s_2(p^{-1}) \geq l_s(p)l_s(p^{-1}) \geq 1$. Hence, by the above theorem, $s_1 = s_2$, and so the extension must be unique.

□

Theorem 9.4. *Fix numbers* $a, b \in \mathbb{R}$ *with* $0 < a < 1 < b$, *and a point* $\omega \in \beta\mathbb{N}$. *Look at the state* $s_\omega : \mathcal{D} \to \mathbb{C}$ *(evaluation at* ω*). Then the followings are equivalent.*

(1) s_ω *extends uniquely to a state on* $\mathcal{B}(\ell^2(\mathbb{N}))$.
(2) *Given* $\varepsilon > 0$, *and* $0 \leq P \in \mathcal{B}(\ell^2(\mathbb{N}))$ *invertible, there exist* $A \in \mathcal{U}_\omega$, $c, d > 0$ *with*
   - $1 - \varepsilon < cd$,
   - $cP_A \leq P_A P P_A$,
   - $dP_A \leq P_A P^{-1} P_A$.
(3) *Given* $\varepsilon > 0$, *and* $P \in \mathcal{H}[a,b]$, *there exist* $A \in \mathcal{U}_\omega$, $c, d > 0$ *with*
   - $1 - \varepsilon < cd$,
   - $cP_A \leq P_A P P_A$,
   - $dP_A \leq P_A P^{-1} P_A$.

Proof. 1) ⇒ 2): Let $\varepsilon > 0$ and take $0 \leq P \in \mathcal{B}(\ell^2(\mathbb{N}))$ invertible. By our previous Theorem we know that if s_ω extends uniquely,

$$l_{s_\omega}(P)l_{s_\omega}(P^{-1}) \geq 1.$$

By property of supremums we may find $D_1 \leq P$ and $D_2 \leq P^{-1}$ with

$$1-\varepsilon/2 \leq s_\omega(D_1)s_\omega(D_2).$$

Now pick numbers $c < s_\omega(D_1)$ and $d < s_\omega(D_2)$, with $1-\varepsilon < cd$. Denoting by f_1 and f_2 the continuous functions on $\beta\mathbb{N}$ corresponding to D_1 and D_2 respectively, choose an open neighborhood $U \in \mathcal{N}_\omega$ where $f_1 > c$ and $f_2 > d$ on U. Consequently, $c\chi_U \leq f_1\chi$ and $d\chi_U \leq f_2\chi$. Set $A = U \cap \mathbb{N} \in \mathcal{U}_\omega$, so

$$cP_A \leq P_A D_1 = P_A D_1 P_A, \text{ and } dP_A \leq P_A D_2 = P_A D_2 P_A,$$

but $D_1 \leq P$ and $D_2 \leq P^{-1}$, so in conjunction with above and by pre and post multiplying by P_A we obtain

$$\begin{aligned} cP_A &\leq P_A D_1 P_A \leq P_A P P_A, \\ dP_A &\leq P_A D_2 P_A \leq P_A P^{-1} P_A. \end{aligned}$$

2) $\Rightarrow$ 3): This is trivial.

3) $\Rightarrow$ 1): Let $s_1, s_2 : \mathcal{B}(\ell^2(\mathbb{N})) \to \mathbb{C}$ be two states that extend s_ω. Given any $\varepsilon > 0$ and $P \in \mathcal{H}[a,b]$, 3) says that we can find $A \in \mathcal{U}_\omega$ and numbers $c, d > 0$ which obey those three bullets. We know that

$$s_1(P_A) = s_2(P_A) = s_\omega(P_A) = 1,$$

and therefore,

$$\begin{aligned} c &= s_1(cP_A) \leq s_1(P_A P P_A) = s_1(P), \\ d &= s_2(dP_A) \leq s_2(P_A P^{-1} P_A) = s_2(P^{-1}). \end{aligned}$$

Thus, $s_1(P)s_2(P^{-1}) \geq cd > 1-\varepsilon$. Since $\varepsilon > 0$ was arbitrary, $s_1(P)s_2(P^{-1}) \geq 1$. Now this can be done for any $P \in \mathcal{H}[a,b]$, so by a previous result $s_1 = s_2$. □

Theorem 9.5. *Fix numbers $0 < a < 1 < b$. The followings are equivalent*

(1) *Kadison-Singer is true.*
(2) *Given $\varepsilon > 0$, and $0 \leq P \in \mathcal{B}(\ell^2(\mathbb{N}))$ invertible, there is a $K \in \mathbb{Z}^+$ with a partition $A_1 \cup \ldots \cup A_K = \mathbb{N}$, and positive sequences $\{c_j\}_{j=1}^K$, $\{d_j\}_{j=1}^K$ satisfying*
   - $1-\varepsilon < c_j d_j$,
   - $c_j P_{A_j} \leq P_{A_j} P P_{A_j}$,
   - $d_j P_{A_j} \leq P_{A_j} P^{-1} P_{A_j}$.
(3) *Given $\varepsilon > 0$, and $P \in \mathcal{H}[a,b]$, there is a $K \in \mathbb{Z}^+$ with a partition $A_1 \cup \ldots \cup A_K = \mathbb{N}$, and positive sequences $\{c_j\}_{j=1}^K$, $\{d_j\}_{j=1}^K$ satisfying*
   - $1-\varepsilon < c_j d_j$,
   - $c_j P_{A_j} \leq P_{A_j} P P_{A_j}$,
   - $d_j P_{A_j} \leq P_{A_j} P^{-1} P_{A_j}$.

(4) *Given $\varepsilon > 0$, there is a $K \in \mathbb{Z}^+$ so that for each $P \in \mathcal{H}[a,b]$, there is a partition $A_1 \cup \ldots \cup A_K = \mathbb{N}$, and positive sequences $\{c_j\}_{j=1}^K$, $\{d_j\}_{j=1}^K$ satisfying*
- $1 - \varepsilon < c_j d_j$,
- $c_j P_{A_j} \leq P_{A_j} P P_{A_j}$,
- $d_j P_{A_j} \leq P_{A_j} P^{-1} P_{A_j}$.

Note how 4) differs from 3). In 3), our epsilon and P are fixed, and out comes this integer K, whereas in 4) only epsilon is fixed and we get a K that works uniformly for all P.

Proof. 1) ⇒ 2): Given $\varepsilon > 0$, and $0 \leq P \in \mathcal{B}(\ell^2(\mathbb{N}))$ invertible the previous Theorem ensures that for each $\omega \in \beta\mathbb{N}$, there is an $B_\omega \in \mathcal{U}_\omega$ (where $B_\omega = U_\omega \cap \mathbb{N}$ for $U_\omega \in \mathcal{N}_w$) and positive numbers $c_\omega, d_\omega > 0$ which satisfy

$$\begin{aligned} c_\omega d_\omega &> 1 - \varepsilon \\ c_\omega P_{B_\omega} &\leq P_{B_\omega} P P_{B_\omega} \\ d_\omega P_{B_\omega} &\leq P_{B_\omega} P^{-1} P_{B_\omega}. \end{aligned}$$

The open cover $\{U_\omega\}_{\omega \in \beta\mathbb{N}}$ of $\beta\mathbb{N}$ must have a finite subcover $\{U_{\omega_l}\}_{l=1}^L$. Out of the sets $\{B_{\omega_l}\}$, whose union is $\mathbb{N}$, make disjoint sets $\{A_j\}_{j=1}^K$ where for each j, $A_j \subseteq B_{\omega_l}$ for some l, and whose disjoint union is also $\mathbb{N}$. In the case where $A_j \subseteq B_{\omega_l}$, we will have

$$c_{\omega_l} P_{A_j} \leq c_{\omega_l} P_{\omega_{B_l}} \leq P_{\omega_{B_l}} P P_{\omega_{B_l}},$$

which implies

$$c_{\omega_l} P_{A_j} P_{A_j} P_{A_j} \leq P_{A_j} [P_{\omega_{B_l}} P P_{\omega_{B_l}}] P_{A_j} = P_{A_j} P P_{A_j}.$$

Therefore, $c_{\omega_l} P_{A_j} \leq P_{A_j} P P_{A_j}$. In similar fashion $d_{\omega_l} P_{A_j} \leq P_{A_j} P^{-1} P_{A_j}$, and 2) holds.

2) ⇒ 3): This is clear.

4) ⇒ 3): Also clear.

4) ⇒ 1): Let $\omega \in \beta\mathbb{N}$ and $\varepsilon > 0$ and $P \in \mathcal{H}[a,b]$ be given. If 4) holds then each $A_j = U_j \cap \mathbb{N}$, with

$$\bigcup_{j=1}^K U_j = \beta\mathbb{N}.$$

Now for some $1 \leq k \leq K$, $\omega \in U_k$, so

$$\begin{aligned} c_k P_{A_k} &\leq P_{A_k} P P_{A_k} \\ d_k P_{A_k} &\leq P_{A_k} P^{-1} P_{A_k} \\ 1-\varepsilon &< c_k d_k. \end{aligned}$$

So by the previous Theorem s_ω extends uniquely. But ω was arbitrary, so Kadison-Singer holds.

3) $\Rightarrow$ 4): Assume 3) true but 4) not true. Then there is an $\varepsilon > 0$ and sequences $P_n \subseteq \mathcal{H}[a,b]$, $(K_n) \subseteq \mathbb{Z}^+$ such that each P_n satisfies the conditions in 3) for the integer K_n but not $K_n - 1$,(basically, K_n is the smallest integer for which the conditions in 3) hold for P_n), and $K_n \nearrow \infty$. Now let

$$P = U_\phi^*(P_1 \oplus P_2 \oplus \ldots)U_\phi,$$

where the

$$U_\phi : \ell^2(\mathbb{N}) \longrightarrow \bigoplus_{i=1}^{\infty} \ell^2(\mathbb{N}_i)$$

is our famous unitary corresponding to the bijective map $\phi : \bigcup_{i=1}^\infty \mathbb{N}_i \to \mathbb{N}$. Now for each n, $\sigma(P_n) \subseteq [a,b]$ so when taking a direct sum and unitary equivalence we get $P \in \mathcal{H}[a,b]$. Therefore for this P we can do some paving for some K, but that K would actually pave all the P_n, a contradiction. $\square$

Definition 9.6. *Given $T \in \mathcal{B}(\ell^2(\mathbb{N}))$ and its corresponding matrix $T = (t_{ij})_{i,j\in\mathbb{N}}$.*

1. *We call T **upper triangular** if $t_{ij} = 0$ for $i > j$, and denote by $\mathcal{T}(\mathbb{N})$ the set of all upper triangular matrices.*
2. *T is said to be **strictly upper triangular** if $t_{ij} = 0$ for $i \geq j$, and we write $\mathcal{T}_0(\mathbb{N})$ for the set of strictly upper triangular matrices.*

A few facts:

1. $\mathcal{T}(\mathbb{N}) \subseteq \mathcal{B}(\ell^2(\mathbb{N}))$ is a subalgebra. This is straightforward.
2. This is a little surprising, $\{T + T^* \ ; \ T \in \mathcal{T}_0(\mathbb{N})\} \subsetneqq \mathcal{H}_0$. A famous example is the matrix $A = (a_{i,j})_{i,j\in\mathbb{N}}$, where $a_{i,i} = 0$ and for $i \neq j, a_{i,j} = \frac{1}{i-j}$. This skew-symmetric matrix gives rise to a bounded operator on $\ell^2(\mathbb{N})$, but its triangular truncation, $T = (t_{i,j})$ defined by $t_{i,j} = a_{i,j}, i < j$, and $t_{i,j} = 0, i \geq j$, is not bounded. Thus, $H = iA$ is Hermitian and if $H = T_1 + T_1^*$, with $T_1 \in \mathcal{T}(\mathbb{N})$, then necessearily $T_1 = iT + D$, with D diagonal. Which shows that H can not be written as a sum of an upper triangular operator and its adjoint.
3. If P is positive and invertible, then there exists an invertible upper triangular operator T such that $P = T^*T$. This can be shown by using the famous algorithm for LU-decomposition for finite matrices and showing that it converges.

(4) Let $T \in \mathcal{T}(\mathbb{N})$ be invertible. Then inverse of T is also in $\mathcal{T}(\mathbb{N})$. Morover if write $T = D + T_0$, where D is a diagonal operator and T_0 is a stricly upper triangular operator, then necessarily D is invertible and $T^{-1} = D^{-1} + T_1$ for some strictly upper triangular operator T_1.

For proofs of these facts see [**?**].

Theorem 9.7 (P-R)**.** *Let $w \in \beta\mathbb{N}$ and $s_w : \mathcal{D} \to \mathbb{C}$. TFAE*

(1) *s_w extends uniquely to a state on $B(\ell^2(\mathbb{N}))$.*
(2) *For all $T \in \mathcal{T}_0(\mathbb{N})$, $l_{s_w}(T + T^*) = 0$.*
(3) *For all $T \in \mathcal{T}_0(\mathbb{N})$ and $\epsilon > 0$ there exists $A \in \mathcal{U}_w$ s.t.*

$$-\epsilon P_A \leq P_A(T + T^*)P_A \leq \epsilon P_A.$$

(4) *For all $T \in \mathcal{T}_0(\mathbb{N})$ and $\epsilon > 0$ there exists $A \in \mathcal{U}_w$ s.t. $\|P_A T P_A\| \leq \epsilon$.*

Proof. (1) $\Rightarrow$ (2) Let s be the unique extension then $s(X) = s(E(X))$. So for all $T \in \mathcal{T}_0(\mathbb{N})$, $s(T + T^*) = s(T) + \overline{s(T)} = 0$ and hence $l_{s_w}(T + T^*) = 0$.

(2) $\Rightarrow$ (1) Let $T \in \mathcal{T}_0(\mathbb{N})$. Since $-T$ is also in $\mathcal{T}_0(\mathbb{N})$, $l_{s_w}((-T) + (-T)^*) = 0$ equivalently $-u_{s_w}(T + T^*) = 0$. So for all $T \in \mathcal{T}_0(\mathbb{N})$, $T + T^*$ is in $\mathcal{U}_w$, the uniqueness domain of s_w. This also means that $(iT) + (iT)^*$ is in $\mathcal{U}(w)$, equivalently, $i(T - T^*) \in \mathcal{U}(w)$. We know that $\mathcal{U}_w$ is a linear subspace. Since both real and imaginary part of T are in $\mathcal{U}(w)$, T is in $\mathcal{U}_w$ for all $T \in \mathcal{T}_0(\mathbb{N})$. Note that this also means $\mathcal{T}(\mathbb{N}) \subseteq \mathcal{U}_w$ since an upper triangular operator can be written as sum of two operators, namely the diagonal part and strict upper part, which belong to uniqueness domain. Now let s_1 and s_2 be two positive extensions of s_w. Let $P \in B(\ell^2(\mathbb{N}))$ be positive and invertible. Then there exists invertible T in $\mathcal{T}(\mathbb{N})$ such that $P = T^*T$ which also means that $P^{-1} = T^{-1}(T^*)^{-1}$. Now by using the facts above we obtain

$$\begin{aligned} s_1(P)s_2(P^{-1}) &= s_1(T^*T)s_2(T^{-1}(T^*)^{-1}) \\ &\geq |s_1(T)|^2 \, |s_2(T^{-1})|^2 \\ &= |s_1(T)s_2(T^{-1})|^2 \\ &= |s_w(E(T))s_w(E(T)^{-1})| = 1. \end{aligned}$$

Since P is arbitrary, $s_1 = s_2$ by Theorem **??**. □

Theorem 9.8. *The following are equivalent.*

(1) *Kadison-Singer is true.*
(2) *For each $\epsilon > 0$ and for each $T \in \mathcal{T}_0(\mathbb{N})$ there exists a k-partition $\{A_i\}_{i=1}^k$ of $\mathbb{N}$ such that $\|P_{A_i} T P_{A_i}\| \leq \epsilon\|T\|$.*
(3) *For every $\epsilon > 0$ there exists $k \in \mathbb{N}$ such that if $T \in \mathcal{T}_0(\mathbb{N})$ then there exists a k-partition $\{A_i\}_{i=1}^k$ of $\mathbb{N}$ such that $\|P_{A_i} T P_{A_i}\| \leq \epsilon\|T\|$.*
(4) *There exists $0 < r < 1$ such that for any $T \in \mathcal{T}_0(\mathbb{N})$ there exists a k-partition $\{A_i\}_{i=1}^k$ of $\mathbb{N}$ such that $\|P_{A_i} T P_{A_i}\| \leq r\|T\|$.*

We will not prove this theorem.

Anderson-Akemann Paving Conjectures. Let $P = (p_{ij}) \in B(\ell^2(\mathbb{N}))$ be an orthogonal projection ($P^2 = P = P^*$). We define $\delta(P) = \sup\{p_{ii} : i = 1, 2, 3, ...\}$.

A-A Conjecture 1. For each $P = P^* = P^2$ there exists $S \in \mathcal{D}$ with $S^2 = I$ such that $\|PSP\| \leq 2\delta(P)$.

A-A Conjecture 2. There exists $0 < \gamma \leq 1/2$ such that if $\delta(P) < \gamma$ then there exists $S \in \mathcal{D}$ with $S^2 = I$ such that $\|PSP\| < 1$.

Anderson and Akemann proved that A-A Conj. 1 $\Rightarrow$ A-A Conj. 2 $\Rightarrow$ K-S is true. But later CEKPT showed that A-A Conjecture 1 is false.

10. Introduction to Group C^*algebras and Crossed Products

10.1. **Unitary Representations and Group Algebras for Discrete Groups.** .
Let G be a group, $\mathcal{H}$ be a Hilbert space, $\mathcal{U}(\mathcal{H})$ be the group of unitaries in $\mathcal{H}$. The homomorphism $\pi : G \to \mathcal{U}(\mathcal{H})$, $\pi(e) = I$, $\pi(gh) = \pi(g)\pi(h)$, $\pi(g^{-1}) = \pi(g)^{-1} = \pi(g)^*$, is called **a unitary representation of** G **on** $\mathcal{H}$.
Let $U_g = \pi(g)$, and consider the $span\{U_g : g \in G\}$.
Let $A, B \in span\{U_g : g \in G\}$, i.e. $A = \sum_{g\in G}^{finite} \lambda_g U_g, \quad B = \sum_{h\in G}^{finite} \mu_h U_h$,

$$\Rightarrow A \cdot B = \sum_{g,h\in G} \lambda_g \mu_h U_{gh} \in span\{U_g : g \in G\},$$

and

$$A^* = \sum_{g\in G} \bar{\lambda}_g U_g^* = \sum \bar{\lambda}_g U_{g^{-1}} \in span\{U_g : g \in G\},$$

i.e. the $span\{U_g : g \in G\}$ is a *-algebra, i.e. an algebra and an operator system.

Letting $\tilde{g} = gh$, $h = g^{-1}\tilde{g}$, $g = \tilde{g}h^{-1}$, we can rewrite the product

$$A \cdot B = \sum_{g,h\in G} \lambda_g \mu_h U_{gh} = \sum_{\tilde{g}} \left(\sum_{g} \lambda_g \mu_{g^{-1}\tilde{g}} \right) U_{\tilde{g}} = \sum_{\tilde{g}} \left(\sum_{h} \lambda_{\tilde{g}h^{-1}} \mu_h \right) U_{\tilde{g}}.$$

This motivates the definition of a group algebra.
Recall the free vector spaces: Let X be any set, then **the free vector space over** X**,** $\mathbb{C}(X)$ is just a vector space with basis given by the elements of X, i.e. one defines

$$\mathbb{C}(X) = \Big\{ \sum_{x\in X, finite} \lambda_x x : \lambda_x \in \mathbb{C} \Big\},$$

and

$$\sum \lambda_x x + \sum \mu_x x = \sum (\lambda_x + \mu_x) x, \quad \lambda\Big(\sum \lambda_x x\Big) = \sum \Big(\lambda\lambda_x\Big) x.$$

Alternatively, one can define $\mathbb{C}(X)$ as a vector space of finitely supported functions,

$$\mathbb{C}(X) = \Big\{ f : X \to \mathbb{C} | \text{ support}(f) \text{ is finite subset} \Big\}.$$

Let $\delta_x : X \to \mathbb{C}, \quad \delta_x(y) = \begin{cases} 1, & x = y \\ 0, & x \neq y \end{cases}$, then $\{\delta_x\}$ is a basis for the space of finitely supported functions. The following correspondence takes place

$$\sum \lambda_x x \longleftrightarrow \sum \lambda_x \delta_x \longleftrightarrow f \text{ with } f(x) = \lambda_x.$$

Definition 10.1. *Let G be a given (discrete) group, then the vector space $\mathbb{C}(G)$ together with the product and *-operation given by*

$$\Big(\sum_{g, finite} \lambda_g \cdot g \Big) \Big(\sum_{h, finite} \mu_h \cdot h \Big) = \sum_{g,h} \lambda_g \mu_h \cdot (gh), \quad \Big(\sum_{g, finite} \lambda_g \cdot g \Big)^* = \sum_g \bar{\lambda}_g \cdot g^{-1},$$

respectively, is called **the group $*$-algebra**.

Note: If $e \in G$ is the unit element, then $1 \cdot e \in \mathbb{C}(G)$ is the unit element of the algebra, hence δ_e is the identity of $\mathbb{C}(G)$.

Alternatively, identifying $f_1 \leftrightarrow \sum \lambda_g \cdot g$, $f_2 \leftrightarrow \sum \mu_h \cdot h$ induces a product

$$f_1 * f_2 = \Big(\sum_g \lambda_g \delta_g \Big) \cdot \Big(\sum_h \mu_h \delta_h \Big) = \sum_{g,h} \lambda_g \mu_h \cdot \delta_{gh}.$$

As a function,

$$(f_1 * f_2)(\tilde{g}) = \sum_g \lambda_g \mu_{g^{-1}\tilde{g}} = \sum_g f_1(g) f_2(g^{-1}\tilde{g}) = \sum_h f_1(\tilde{g}h^{-1}) f_2(h).$$

This product is called **the convolution of functions.**
Also, note that $(f_1^*)(g) = \overline{f_1(g^{-1})}$.
Let $\pi : G \to \mathcal{U}(\mathcal{H})$ be a unitary representation, which, clearly, can be extended to a linear map $\tilde{\pi} : G \to B(\mathcal{H})$, $\tilde{\pi}(\sum_g \lambda_g \cdot g) = \sum_g \lambda_g \pi(g)$.
Also, having

$$\tilde{\pi}\Big((\sum_g \lambda_g g)(\sum_h \mu_h h) \Big) = \tilde{\pi}\Big(\sum_{g,h} (\lambda_g \mu_h) gh \Big) = \sum_{g,h} (\lambda_g \mu_h) \pi(gh)$$

$$= \sum_{g,h} \lambda_g \mu_h \pi(g) \pi(h) = \Big(\sum_g \lambda_g \pi(g) \Big) \Big(\sum_h \mu_h \pi(h) \Big) = \tilde{\pi}\Big(\sum_g \lambda_g g \Big) \tilde{\pi}\Big(\sum_h \mu_h h \Big),$$

and

$$\tilde{\pi}\Big((\sum_g \lambda_g g)^* \Big) = \tilde{\pi}\Big(\sum_g \bar{\lambda}_g g^{-1} \Big) = \sum_g \bar{\lambda}_g \pi(g^{-1})$$

$$= \sum_g \bar{\lambda}_g \pi(g)^* = \Big(\sum_g \lambda_g \pi(g) \Big)^* = \tilde{\pi}\Big((\sum_g \lambda_g) \Big)^*,$$

gives that the extension $\tilde{\pi}$ is a *-homomorphism.
Conversely, given $\tilde{\pi} : \mathbb{C}(G) \to B(\mathcal{H})$ a unital *-homomorphism, setting $\pi(g) = \tilde{\pi}(1 \cdot g)$ defines a unitary representation of G.

Proposition 10.2. *$\pi \leftrightarrow \tilde{\pi}$ is an one-to-one correspondence between unitary representations of G and unital *-homomorphisms of $\mathbb{C}(G)$.*

Definition 10.3. *The full group C^* algebra denoted as $C^*(G)$, (or $C^*_f(G)$) is the completion of $\mathbb{C}(G)$ under the norm defined as follows: Let $a = \sum \lambda_g \cdot g \in \mathbb{C}(G)$, define $\|a\| = \sup\{\|\tilde{\pi}\Big(\sum \lambda_g \cdot g\Big)\| : \forall \pi$ unitary representations$\}$, then for any $a \in \mathbb{C}(G)$, one shows*

- $\|a^*a\| = \|a\|^2$,
- $\|a\| \neq 0$ *for* $a \neq 0$,
- $\|\cdot\|$ *is really a norm.*

Why finite? Let $a = \sum \lambda_g \cdot g \in \mathbb{C}(G)$, then

$$\|\tilde{\pi}(a)\| = \|\sum_{g,finite} \lambda_g \pi(g)\| \leq \sum_{g,finite} |\lambda_g| \cdot \underbrace{\|\pi(g)\|}_{=1,unitary} = \sum_{g,finite} |\lambda_g| < +\infty,$$

therefore we get $\|a\| \leq \sum |\lambda_g| < +\infty$.

Examples of C*(G)

1. Group of integers $\mathbb{Z}$: Let $\pi : \mathbb{Z} \to B(\mathcal{H})$ be a homomorphism with $\pi(0) = I, \pi(1) = U$, unitary. Let $a = \sum \lambda_n n \in \mathbb{C}(\mathbb{Z})$, then $\tilde{\pi}(a) = \sum \lambda_n U^n$.
Look at $\|\tilde{\pi}(a)\|$ =sup$\{|\sum \lambda_n e^{in\theta}| : e^{in\theta} \in \sigma(U)\}$ which forces
$\|a\|_{C^*(\mathbb{Z})} = \|\sum \lambda_n z^n\|_{C(\mathbb{T})}$.
Therefore, $C^*(\mathbb{Z}) \cong C(\mathbb{T})$.

2. $\mathbb{Z}_2 = \{[0], [1]\}$: Here we let $\pi_2 :\to B(\mathcal{H}), \pi_2([0]) = I, \pi_2([1]) = U$-unitary with $U^2 = I$.
$\sigma(U) \subseteq \{\pm 1\}$ and the most general such unitary will be $U \cong \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$. Any element a $\in \mathbb{Z}_2$, a=$\lambda_0[0] + \lambda_1[1]$, $\tilde{\pi}(a) = \lambda_0 I + \lambda_1 U = \begin{pmatrix} \lambda_0 + \lambda_1 & 0 \\ 0 & \lambda_0 - \lambda_1 \end{pmatrix}$.
$\therefore \|\tilde{\pi}(a)\| = \max\ \{|\lambda_0 + \lambda_1|, |\lambda_0 - \lambda_1|\}$
$\Rightarrow \|a\|_{C^*(\mathbb{Z}_2)} = \max\ \{|\lambda_0 + \lambda_1|, |\lambda_0 - \lambda_1|\}$
Therefore, $C^*(\mathbb{Z}_2) \cong \mathbb{C} \oplus \mathbb{C}$ under the mapping
$\lambda_0[0] + \lambda_1[1] \longmapsto (\lambda_0 + \lambda_1, \lambda_0 - \lambda_1)$.

3. $\mathbb{Z}^2 = \{(m, n) : m, n \in \mathbb{Z}\}$: Here $\pi : \mathbb{Z} \to B(\mathcal{H}), \pi((0, 0)) = I, \pi((1, 0)) = U, \pi((0, 1)) = V$, U and V commuting unitaries since $\mathbb{Z}^2$ is an abelian group.
$\pi((m, n)) = U^m V^n$, any element of $\mathbb{Z}^2$, a=$\sum \lambda_{m,n}(m, n)$ then $\tilde{\pi}(a) = \sum \lambda_{m,n} U^m V^n$.
U and V are commuting unitaries therefore, $C^*\{U, V\} \cong C(X)$. Suppose U corresponds to a function f_1 and V to f_2.
We Know: 1). $C(X) = C^*(f_1, f_2)$. We claim that $x \longmapsto (f_1(x), f_2(x))$ is

1-1.
Let x,y $\in X$, $(f_1(x), f_2(x)) = (f_1(y), f_2(y))$ which implies that every function in $C^*\{f_1, f_2\}$ is equal at x and y. Therefore by Urysohn's lemma, x=y which establishes the claim.
2) We have $U^* = U^{-1} \Rightarrow \overline{f_1(x)} = f_1(x)^{-1} \Rightarrow f_1(x) \in \mathbb{T}$.
Similarly $V^* = V^{-1} \Rightarrow f_2(x) \in \mathbb{T}$.

$$\begin{aligned}\therefore \|\tilde{\pi}(a)\| &= \|\sum \lambda_{m,n} U^m V^n\| = \|\sum \lambda_{m,n} f_1(.)^m f_2(.)^n\|_{C(X)} \\ &= sup\{|\sum \lambda_{m,n}\alpha^m\beta^n| : (\alpha = f_1(x), \beta = f_2(x)) \in \mathbb{T}\}.\end{aligned}$$

$$\|a\| = \|\sum \lambda_{m,n}(m,n)\|_{C^*(\mathbb{Z}^2)} = sup\{|\sum \lambda_{m,n}\alpha^m\beta^n| : (\alpha,\beta) \in \mathbb{T}^2\} = \|\sum \lambda_{m,n} z_1^m z_2^n\|_{C(\mathbb{T}^2)}.$$

$\therefore C^*(\mathbb{Z}^2) \cong C(\mathbb{T}^2)$.

4. **Non Abelian group, $\mathbb{F}_2$ -free group on 2 generators**: Let the generators be g_1, g_2 then a typical element w $\in \mathbb{F}_2$ is often called a **word** in g_1, g_2 and written as $w = g_{i_1}^{n_1} g_{i_2}^{n_2} ... g_{i_p}^{n_p}$ where $i_l \neq i_{l+1}, n_l \in \mathbb{Z}$.
Let $v \in \mathbb{F}_2, v = g_{j_1}^{m_1} g_{j_2}^{m_2} ... g_{j_q}^{m_q}$ where $j_l \neq j_{l+1}, m_q \in \mathbb{Z}$ and the identity of the group e is any word where all $n_i = 0$.
$v.w = g_{j_1}^{m_1} g_{j_2}^{m_2} ... g_{j_q}^{m_q} . g_{i_1}^{n_1} g_{i_2}^{n_2} ... g_{i_p}^{n_p}$ if $j_q \neq i_1$ called concatenation and equals $g_{j_1}^{m_1} g_{j_2}^{m_2} ... g_{j_{q-1}}^{m_{q-1}} .(g_{i_1}^{m_q+n_1}) g_{i_2}^{n_2} ... g_{i_p}^{n_p}$ if $j_q = i_1$. $w^{-1} = g_{i_p}^{-n_p} ... g_{i_1}^{-n_1}$.

Universal property: Given any group G and $h_1, h_2 \in G$ $\exists$! group homomorphism $\pi : \mathbb{F}_2 \to G$ via $g_1 \longmapsto h_1$ and $g_2 \longmapsto h_2$. is called the universal property.

Let $\pi : \mathbb{F}_2 \to G$ be a unitary representation $\Leftrightarrow \pi(g_1) = U_1$ and $\pi(g_2) = U_2$.
Let $a \in \mathbb{C}(\mathbb{F}_2), a = \sum \lambda_w . w$
$\tilde{\pi}(a) = \sum \lambda_w . \pi(w) \Rightarrow \tilde{\pi}(a) = \sum \lambda_w U_{i_1}^{n_1} ... U_{i_p}^{n_p}$.
$\therefore \|a\|_{C^*(\mathbb{F}_2)}$ =sup$\{\|\sum \lambda_w \pi(w)\|$: all pairs of unitaries $U_1, U_2\}$ is sometimes referred as “Non Commutative Torus”.

10.2. **The Left Regular Representation.**
Let G be a discrete group, look at $l^2(G)$.

$$l^2(G) = \{f : G \to \mathbb{C} : \sum_{g \in G} |f(g)|^2 < \infty\} = \{\sum_{g \in G} \lambda_g e_g : \sum |\lambda_g|^2 < \infty\}$$

where $\{e_g : g \in G\}$ is an orthonormal basis.
Inner Product: $< f_1, f_2 >_{l^2(G)} = \sum_{g \in G} f_1(g)\overline{f_2(g)}$
$< \sum \lambda_g e_g, \sum \mu_g e_g > = \sum \lambda_g \bar{\mu_g}$.
Here e_g corresponds to δ_g therefore, $\sum \lambda_g e_g$ corresponds to a function f

where f(g)=λ_g. The **left regular representation of G,** is the group representation $\lambda : G \to B(l^2(g))$ where $\lambda(g)$ is the unitary uniquely defined by $\lambda(g)e_h = e_g h$.
Note: $gh_1 = gh_2 \Leftrightarrow h_1 = h_2, \lambda(g)e_{g^{-1}h} = e_h$, each $\lambda(g)$ is just a permutation of the basis vector and $\lambda(g)(\sum \lambda_h e_h) = \sum \lambda_h e_g h$.
¿From the function view point, $\lambda(g)\delta_h = \delta_g h$
$(\lambda(g)\delta_h)(k) = \delta_{gh}(k) = \delta_h(g^{-1}k)$.
$(\lambda(g)f)k = f(g^{-1}k)$.
The **reduced C*algebra of G** is denoted $C^*_\lambda(G) = C^*_r(G) = C^*(\{\lambda(g) : g \in G\}) \subseteq B(l^2(G))$, $C^*(G) = \overline{\{\sum \alpha_g \lambda(g)\}}^{\|\cdot\|}$.

We define the group von Neumann by $VN(G) := \{\lambda(g) : g \in G\}'' \subseteq B(\ell^2(G))$. An application on von Neumann's double commutant theorem shows that $VN(G) = C^*_\lambda(G)'' = WOT - closure C^*_\lambda(G)$.

We have not yet shown that the seminorm defined on $\mathbb{C}(G)$ is actually a norm. The reduced representation shows that $\|\cdot\|$ is actually a norm on $\mathbb{C}(G)$. To see this let $a = \sum_{g\in G} \alpha_g g \in \mathbb{C}(G)$ and let $\tilde{\lambda}(a) = \sum_{g \in G} \alpha_g \lambda(g) \in B(\ell^2(G))$. We have $\tilde{\lambda}(g)e_e = \sum_{g\in G} \alpha_g \lambda(g) e_e = \sum_{g\in G} e_g$. Hence $\left\|\tilde{\lambda}(a)\right\| \geq \left\|\tilde{\lambda}(g)e_e\right\| = \left\|\sum_{g\in G}\alpha_g e_g\right\| = \left(\sum_{g\in G}|\alpha_g|^2\right)^{1/2} \neq 0$ for $a \neq 0$. This yields the estimate,

$$\left(\sum_{g\in G}|\alpha_g|^2\right)^{1/2} \leq \left\|\tilde{\lambda}(a)\right\| \leq \|a\|_{C^*(G)} \leq \sum_{g\in G}|\alpha_g|$$

We will now look at some examples of $C^*_\lambda(G)$.
Example 8 Let $G = \mathbb{Z}$ and note that $\ell^2(\mathbb{Z}) = span\{e_n : n \in \mathbb{Z}\}$ and let $B = \lambda(1)$. Note that $Be_n = \lambda(1)e_n = e_{n+1}$ and $\lambda(n) = B^n$. The operator B is called the bilateral shift on $\ell^2(\mathbb{Z})$. Every $A \in B(\ell^2(\mathbb{Z}))$ has a matrix $A = (a_{i,j})$. The matrix is infinite in both directions and when we write our matrices we distinguish the $(0,0)$ entry by drawing a box around,

$$A = \begin{bmatrix} \ddots & \vdots & \vdots & \vdots & \\ \cdots & a_{-1,-1} & a_{-1,0} & a_{-1,1} & \cdots \\ \cdots & a_{0,-1} & \boxed{a_{0,0}} & a_{0,1} & \cdots \\ \cdots & a_{1,-1} & a_{1,0} & a_{1,1} & \cdots \\ & \vdots & \vdots & \vdots & \ddots \end{bmatrix}$$

For example since $Be_n = e_{n+1}$ we see that the matrix of B is

$$B = \begin{bmatrix} \ddots & \ddots & \ddots & \ddots & & \\ \ddots & 1 & 0 & 0 & 0 & \ddots \\ \ddots & 0 & 1 & \boxed{0} & 0 & \ddots \\ \ddots & 0 & 0 & 1 & 0 & \ddots \\ \ddots & 0 & 0 & 0 & 1 & \ddots \\ & \ddots & \ddots & \ddots & \ddots & \end{bmatrix}$$

Let $a = \sum_{n=-N}^{N} \alpha_n n$ and note that $\tilde{\lambda}(a) = \sum_{n=-N}^{N} \alpha_n B^n$ and so the matrix of $\tilde{\lambda}(a)$ has the form

$$\tilde{\lambda}(a) = \begin{bmatrix} \ddots & \ddots & \ddots & \ddots & & \\ \ddots & \alpha_1 & \alpha_0 & \alpha_{-1} & \alpha_{-2} & \ddots \\ \ddots & \alpha_2 & \alpha_1 & \boxed{\alpha_0} & \alpha_{-1} & \ddots \\ \ddots & \alpha_3 & \alpha_2 & \alpha_1 & \alpha_0 & \ddots \\ \ddots & \alpha_4 & \alpha_3 & \alpha_2 & \alpha_1 & \ddots \\ & \ddots & \ddots & \ddots & \ddots & \end{bmatrix}$$

Note that the entries of the matrix are constant on diagonals and so there is a function $\alpha : \mathbb{Z} \to \mathbb{C}$ such that $a_{i,j} = \alpha(i-j)$. A bounded operator on $\ell^2(\mathbb{Z})$ whose matrix is constant on the diagonal is called a Laurent operator.

In order to identify $C^*_\lambda(\mathbb{Z})$ we need the Fourier transform. Let $L^2(\mathbb{T})$ be the usual Lebesgue space of the circle and let us denote $z^n = e^{in\theta}$ Given $f \in L^2(\mathbb{T})$ we write

$$\hat{f}(n) = \langle f, z^n \rangle = \frac{1}{2\pi} \int_0^{2\pi} f(e^{i\theta}) e^{-in\theta}\, d\theta$$

and we write the association as $f \sim \sum_{n=-\infty}^{\infty} \hat{f}(n) z^n$ and let $U : L^2(\mathbb{T}) \to \ell^2(\mathbb{Z})$ by $z^n \mapsto e_n$ and so $f \mapsto \sum_{n=-\infty}^{\infty} \hat{f}(n) e_n$. Now,

$$U^{-1}BUz^n = U^{-1}Be_n = U^{-1}e_{n+1} = z^{n+1} = M_z(z^n)$$

and so $U^{-1}BU = M_z$. Hence,

$$U^{-1}(\sum_{n=-N}^{N} \alpha_n B^n)U = \sum_{n=-N}^{N} \alpha_n B^n = M_{\sum_{n=-N}^{N} \alpha_n z^n}$$

and so $C^*_\lambda(\mathbb{Z})$ is the norm closure of the set of polynomials of the form $\sum_{n=-N}^{N} \alpha_n z^n$ which by the Stone-Weierstrass theorem is $C(\mathbb{T})$. In particular $U^{-1}C^*_\lambda(\mathbb{Z})U = \{M_f \,:\, f \in C(\mathbb{T})\}$.

When $f \in C(\mathbb{T})$ we have $UM_fU^{-1} \in B(\ell^2(\mathbb{Z}))$ and the matrix of this operator is given by

$$M_f = \begin{bmatrix} \ddots & \ddots & \ddots & \ddots & & \\ \ddots & \hat{f}(1) & \hat{f}(0) & \hat{f}(-1) & \hat{f}(-2) & \ddots \\ \ddots & \hat{f}(2) & \hat{f}(1) & \boxed{\hat{f}(0)} & \hat{f}(-1) & \ddots \\ \ddots & \hat{f}(3) & \hat{f}(2) & \hat{f}(1) & \hat{f}(0) & \ddots \\ \ddots & \hat{f}(4) & \hat{f}(3) & \hat{f}(2) & \hat{f}(1) & \ddots \\ & \ddots & \ddots & \ddots & \ddots & \end{bmatrix}$$

We will show that the group von Neumann algebra is $U\mathcal{M}U^{-1} = \{UM_fU^{-1} : f \in L^\infty(\mathbb{T})\}$. Since $\mathcal{M}$ is a MASA, $U\mathcal{M}U^{-1}$ is also a MASA and we see that $U\mathcal{M}U^{-1} = U\mathcal{M}''U^{-1} \supseteq C^*_\lambda(\mathbb{Z})'' = VN(\mathbb{Z})$. If $R \in C^*_\lambda(\mathbb{Z})'$, then $RB = BR$ and this shows that $T = U^{-1}RU$ commutes with multiplication by z. If we set $g = T(1)$ we see that $T(z^n) = gz^n$ for all $n \in \mathbb{Z}$. For any $f \in L^\infty(\mathbb{T})$ we get $\langle TM_fz^i, z^j\rangle = \left\langle T(\sum_{n=-\infty}^{\infty} \hat{f}(n)z^n)z^i, z^j\right\rangle = \sum_{n=-\infty}^{\infty} \hat{f}(n)\langle gz^{n+i}, z^j\rangle = \langle (fg)z^i, z^j\rangle = \langle M_fTz^i, z^j\rangle$. Hence, $T = M_g \in \mathcal{M}'$ and we get $C^*_\lambda(\mathbb{Z})' \subseteq U\mathcal{M}'U^{-1}$ which implies $VN(\mathbb{Z}) \supseteq U\mathcal{M}'U^{-1} = U\mathcal{M}U^{-1}$.

Example 9 Consider the case $G = \mathbb{Z}_2 = \{0, 1\}$. Here $\ell^2(\mathbb{Z}_2)$ can be identified with $\mathbb{C}^2$. Under the representation $\lambda(0) = I_2$ and $\lambda(1) = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$.

A general element of $C^*_\lambda(\mathbb{Z}_2)$ is of the form $\begin{bmatrix} \alpha & \beta \\ \beta & \alpha \end{bmatrix}$ for $\alpha, \beta \in \mathbb{C}$.

Example 10 To identify $C^*_\lambda(\mathbb{Z}^2)$ we can carry out a similar analysis to the one used for $\mathbb{Z}$ and we see that $C^*_\lambda(\mathbb{Z})$ can be identified with $C(\mathbb{T}^2)$ and that the group von Neumann algebra can be identified with $L^\infty(\mathbb{T}^2)$. Once again we can define a matrix for each operator on $A = \ell^2(\mathbb{Z}^2)$. However since $L^2(\mathbb{T}^2)$ is spanned by the orthonormal basis $e_{m,n} = e^{in\theta_1}e^{in\theta_2}$ we see that the matrix is naturally indexed not by pairs of integers but tby 4-tuples.

In general if G is discrete abelian group, then we can define the dual group $\hat{G}$ whcih is the set of all homomorphisms from G into $\mathbb{T}$. This is a group under pointwise product and it has we give it the topology of pointwise convergence. Under these conditions $\hat{G}$ is a compact group. As an example the group dual to $\mathbb{Z}$ is $\mathbb{T}$ where the pairing is given by $\lambda(n) = \lambda^n$. In general for a discrete abelian group G we have $C^*_\lambda(G) = C(\hat{G})$ and $VN(G) = L^\infty(\hat{G})$.

Example 11 We now look at $\mathbb{F}_2$, the free group on two generators u, v. An element $w \in \mathbb{F}_2$ can be thought of as a word in u, v and so the action of $B_w = \lambda(w)$ on e_g is $B_we_g = e_{wg}$. Let $\mathcal{W}_v$ be the set of words that begin with a non-zero power of v and define $\ell^2_n = \text{span}\{e_{u^nw} : w \in \mathcal{W}_v\}$. The direct sum of the spaces ℓ^2_n is $\ell^2(\mathbb{F}_2)$ and B_u maps ℓ^2_n isometrically onto ℓ^2_{n+1}. This gives us a sense of how "large" the Hilbert space $\ell^2(\mathbb{F}_2)$ is.

Comparison between $C^*(G)$ and $C^*_\lambda(G)$: By the definition of $C^*(G)$, if we are given any $\pi : G \longrightarrow \mathcal{B}(\mathcal{H})$ unitary representation and $a \in \mathbb{C}(G)$, then $\|a\|_{C^*(G)} \geq \|\tilde{\pi}(a)\|$. Hence if $\{a_n\} \subseteq \mathbb{C}(G)$ Cauchy in $C^*(G)$, then $\{\tilde{\pi}(a_n)\} \subseteq \mathcal{B}(\mathcal{H})$ is Cauchy. So there is a well defined map $\bar{\pi} : C^*(G) \longrightarrow C^*(\{\pi(g) : g \in G\}) \subseteq \mathcal{B}(\mathcal{H})$. For b_1, $b_2 \in C^*(G)$, there exists $\{a_n\}, \{a'_n\} \subseteq \mathbb{C}(G)$ such that $\|b_1 - a_n\| \to 0, \|b_2 - a'_n\| \to 0$. Thus, $\|b_1 b_2 - a_n a'_n\| \to 0$ which implies $\|\bar{\pi}(b_1 b_2) - \bar{\pi}(a_n a'_n)\| \to 0$. But $\bar{\pi}(a_n a'_n) = \tilde{\pi}(a_n a'_n) = \tilde{\pi}(a_n)\tilde{\pi}(a'_n) \to \bar{\pi}(b_1)\bar{\pi}(b_2)$. Hence, $\bar{\pi}(b_1 b_2) = \bar{\pi}(b_1)\bar{\pi}(b_2)$. Similarly, $\bar{\pi}(b_1^*) = \bar{\pi}(b_1)^*$. so $\bar{\pi}$ is a $*$-homomorphism. In particular, there is always a $*$-homomorphism from $C^*(G)$ onto $C^*_\lambda(G)$.

One of the question that arises is that when is this map one-to-one? When G is an amenable group, this map is one-to-one. Amenable groups include all abelian groups, finite groups. Also, $O \to N \to G \to H \to O$ and N, H amenable implies G amenable.

The free group on two generators, $\mathbb{F}_2$, is not amenable and the map from $C^*(G)$ to $C^*_\lambda(G)$ is not one-to-one.

10.3. **Group Actions.**

Definition 10.4. Groups acting on sets: *Given a group G and a set X. We let $Perm(X) = \{h : X \longrightarrow X | h - invertible\}$. Then, $Perm(X)$ is a group under the operation of composition. We will define an action of G on X in the following three ways:*

1. *An action of G on X is a homomorphism $\alpha : G \longrightarrow Perm(X)$.*
2. *If we let $\alpha(g) = h_g \in Perm(X)$, then $h_{g_1} \circ h_{g_2} = h_{g_1 g_2}$, $h_e = id_X$ where $\{h_g : g \in G\} \subseteq Perm(X)$ is a group.*
3. *Given $g \in G$, $x \in X$, set $g \cdot x = \alpha(g)(x)$. This is a map $G \times X \longrightarrow X$ such that $(g, x) \longrightarrow g \cdot x$ having properties*
   (a) *$e \cdot x = x$ for all $x \in X$*
   (b) *$g_1 \cdot (g_2 \cdot x) = h_{g_1}(g_2 \cdot x) = h_{g_1}(h_{g_2}(x)) = h_{g_1} \circ h_{g_2}(x) = h_{g_1 g_2}(x) = (g_1 g_2) \cdot x$*

Assuming the third definition, if we define $h_g(x) = g \cdot x$ and because $(h_{g^{-1}} \circ h_g)(x) = g^{-1} \cdot (g \cdot x) = e \cdot x = x$, and similarly we can prove that $(h_g \circ h_{g_{-1}})(x) = x$. Thus each h_g is an invertible map and so $h_g \in Perm(X)$.
Note: When X is a toplogical space, by an action of G on X, we mean that each h_g is a homeomorphism.
Example 12 : Let X be a toplogical space. Fix $h : X \longrightarrow X$ homeomorphism and define

$$\alpha(n) = \begin{cases} h^{(n)} = \underbrace{h \circ h \ldots \circ h}_{n-times} & n > 0 \\ (h^{-1})^{|n|} = \underbrace{h^{-1} \circ h^{-1} \ldots \circ h^{-1}}_{|n|-times} & n < 0 \\ id_X & n = 0 \end{cases}$$

Then α defines an action of $\mathbb{Z}$ on X. Let us see few examples of this action:

(1) When $X = \mathbb{R}$, $h(r) = r + 1$, the $h^n(r) = r + n$.
(2) when $X = \mathbb{T}$, fix θ, $0 \leq \theta < 1$, $h(\lambda) = e^{2\pi i\theta}\lambda$, $h^n(\lambda) = e^{2\pi in\theta}\lambda$. If $\theta = \frac{p}{q}$, $h^q(\lambda) = \lambda$ and if θ is an irrational, $h^n(\lambda) \neq \lambda$ for all $n \neq 0$ but given $\epsilon > 0$, there exists $\{n_k\}$ such that $|e^{2\pi in_k\theta} - 1| < \epsilon$ which is equivalent to $|h^{(n_k)}(\lambda) - \lambda| < \epsilon$

Definition 10.5. *Given an action of G on X, a point $x \in X$ is called recurrent if for every neighborhood U of x, there exists $e \neq g \in G$ such that $g \cdot x \in U$*

In the above example when $X = \mathbb{T}$, all points are recurrent.

Definition 10.6. *A point $x \in X$ is non-recurrent if there exists a neighborhood U of x such that $g \cdot x \notin U$ for all $e \neq g \in G$.*

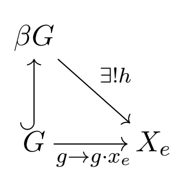
In the above example when $X = \mathbb{R}$, every point is non-recurrent.

Definition 10.7. *A point $x \in X$ is called wandering if there exists a neighborhood U of x such that $(g_1 \cdot U) \cap (g_2 \cdot U) = \phi$ for all $g_1 \neq g_2$.*

Note that wandering implies non-recurrent. When $X = \mathbb{R}$ in Example 12, every point is wandering.
Example 13 (Cayley): Let G be a group and X = G. Define $h_g : G \longrightarrow G$ as $h_g(g^{'}) = gg^{'}$. Clearly h_g is one-to-one and onto, $h_{g_1} \circ h_{g_2} = h_{g_1g_2}$ and $h_e = id_G$. So, $\alpha(g) = h_g$ is a group action.
Example 14 Let G be a discrete group and $X = \beta G$. Fix $g \in G$. By above example and by the properties of βG, there exists a unique continuous map $h_g : \beta G \longrightarrow \beta G$ such that $h_g(g^{'}) = gg^{'}$ for all $g^{'} \in G$. Given $\omega \in \beta G$, since G is dense in βG, there exists a net $\{g^{'}_\lambda\} \subseteq G$ such that $g^{'}_\lambda \to \omega$. Thus, $h_g(\omega) = \lim_\lambda h_g(g^{'}_\lambda) = \lim_\lambda gg^{'}_\lambda$. Now $h_e(g^{'}) = g^{'}$ which implies that $h_e(\omega) = \omega$ for all $\omega \in \beta G$. By uniqueness, $h_e = id_{\beta G}$. Also, $h_{g_1} \circ h_{g_2} = h_{g_1g_2}$ on G and G is dense in βG, so $h_{g_1} \circ h_{g_2} = h_{g_1g_2}$ on βG. Hence G acts on βG denoted by $\omega \to g \cdot \omega$.

11. Dynamical syatems and βG

Utrafilters in dynamics motivates why people in dynamical system are interested in βG and the action of G on βG.
Let X be any compact Hausdroff space with a continuous action.Pick a point $x_e \in X$. To study dynamical propeties of the point X_e only need to look at $\overline{\{g \cdot x_e : g \in G\}} = X_e$, closed orbit.



Since $h(G) = \{g \cdot x_e : g \in G\}$, therefore $h(\beta G) = X_e$.
If $g_\lambda \longrightarrow w \in \beta G$, then $h(gg_\lambda) = gg_\lambda \cdot x_e$. Let $h(w) = x_w.g_\lambda \longrightarrow w$ which implies that $g_\lambda \cdot x_e \longrightarrow x_w$. Therefore $h(gg_\lambda) = g \cdot (g \cdot x_e) \longrightarrow g \cdot x_w$. So, $h(g \cdot w) = g \cdot h(w) = g \cdot x_w$.
This shows that in a sense every dynamical system is a "quotient" of the action of G on βG. In this sense βG is "universal" dynamical system.

The semigroup $\mathbb{N}$
In a similar fashion we fix n and look at the map $\mathbb{N} \longrightarrow \mathbb{N}$ given by $m \longrightarrow n + m$.

$$\begin{array}{ccc} \beta\mathbb{N} & \xrightarrow{\exists ! h_n} & \beta\mathbb{N} \\ \uparrow & & \uparrow \\ \mathbb{N} & \xrightarrow[m \to n+m]{} & \mathbb{N} \end{array}$$

Given $w \in \beta\mathbb{N}$ we write $n + w \equiv h_n(w)$.

The **corona** is defined to be $G^* = \beta G \setminus G$. Note that G is open in βG. This implies that G^* is closed and so compact.

Lemma 11.1. *Let $w \in G^*$ and $g \in G$.Then $g \cdot w \in G^*$.*

Proof. Let $w \in G^*$ and let $g_\lambda \longrightarrow w$.Suppose $g \cdot w = h \in G$. Then $gg_\lambda \longrightarrow h$. Let $U = \{h\}$,open. Then there exists λ_0 such that $gg_\lambda \in U$ for every $\lambda \geq \lambda_0$. Therefore, $gg_\lambda = h$ for each $\lambda \geq \lambda_0$ which implies $g_\lambda = g^{-1}h$ for $\lambda \geq \lambda_0$.So, $g_\lambda \longrightarrow g^{-1}h$ which implies $w = g^{-1}h \in G$.
Hence G^* is G-invariant subset.

□

Back to Kadison-Singer
Recall by **Reid**, we know that if w is a rare ultrafilter then the extension was unique.
Anderson conjectured If w is $\delta-$ stable ultrafilter then the extension is unique.
Natural Question !
What are the dynamical properties of such ultrafilters ?
¿From now in this section we assume that G is a countable group with discrete topology. Recall by a theorem of **Choquet**, w is $\delta-$ stable if and only if w is a P-point.

Proposition 11.2. *Let $w \in G^*$ be $\delta-$ stable. Then w is non-recurrent in G^*.*

Proof. By a theorem of **Veech**, for $w \in G^*$, $g_1 \cdot w = g_2 \cdot w$ if and only if $g_1 = g_2$ (**such an action is called free**). For each g in G, let $U_g = \beta G \setminus \{g \cdot w\}$, open. As $w \neq g \cdot w$, $w \in U_g$.this gives $w \in \cap\{U_g : g \in G, g \neq e\}$ which is a G_δ set. Because w is a P-point, there exists an open set U, $w \in U, U \subseteq \cap\{U_g : g \in G, g \neq e\}$
$\implies g \cdot w \notin U$ for each $g \neq e$.
$\implies w$ is non-recurent.

□

Proposition 11.3. *Let $w \in \mathbb{N}$ be a rare ultrafilter. Then w is wandering in $\mathbb{N}^*$.*

Proof. Write $\mathbb{N} = \{1\} \cup \{2\} \cup \{3,4\} \cup \{5,6\} \cup \{7,8,9\} \cup \{10,11,12\} \cup \cdots$
Let $A = \{1\} \cup \{3,4\} \cup \{7,8,9\} \cup \cdots$
As $\mathcal{U}_w$ is an ultrafiter, therefore either $A \in \mathcal{U}_w$ or $A^c \in \mathcal{U}_w$. Assume $A \in \mathcal{U}_w$. Since w is rare, there exists $B \in \mathcal{U}_w$ such that B intersected with each of the above finite sets has atmost one element. Look at $C = A \cap B \in \mathcal{U}_w$.

Picture:
$\underline{1}$ 2 $\underline{3\ \ 4}$ 5 6 $\underline{7\ \ 8\ \ 9}$ 10 11 12 $\underline{13\ \ 14\ \ 15\ \ 16}\ldots$

C can have atmost 1 element from each of the underlined sets in this picture.

Note, card$\{(n + C) \cap C\} \leq n - 1$ for each n.
Now, $C \in \mathcal{U}_w$, therefore there exists an open neighbourhood U of w in $\beta\mathbb{N}$ such that $C = U \cap \mathbb{N}$.
$\implies$ card$\{(n + U) \cap U \cap \mathbb{N}\} \leq n - 1$ for each n
$\implies (n + U) \cap U$ is a finite set and contained in $\mathbb{N}$.
$\therefore \quad ((n + U) \cap \mathbb{N}^*) \cap (U \cap \mathbb{N}^*) = \emptyset$
$\therefore V = U \cap \mathbb{N}^*$ is an open set in $\mathbb{N}^*$ and $(n + V) \cap V = \emptyset$ for every n.
With a similar argument it can be shown that for every $n \neq j$,
$(n + V) \cap (j + V) = \emptyset$.
This proves that w is a wandering in $\mathbb{N}^*$

□

Corollary 11.4. *Assuming the continuum hypothesis, then $\mathbb{N}^*$ contains a dense set of wandering points.*

Proof. The continuum hypothesis imples that rare ultarfilters exist and are dense in $\mathbb{N}^*$. □

Theorem 11.5 (Davidson)**.** *Let G be a countable discrete group with $\omega \in \beta G$ a rare ultrafilter. Then ω is a wandering point in $G^* = \beta G \backslash G$.*

Proof. Choose an ascending chain of finite subsets of G

$$\{e\} = G_0 \subseteq G_1 \subseteq G_2 \ldots$$

such that $G_n = G_n^{-1}$ and $\bigcup_{n=0}^{\infty} G_n = G$. The former statement just means that each G_n is closed under taking inverses. Now we let

$$P_n = G_n \cdot G_{n-1} \cdots \cdot G_1 \cdot G_0, \text{ for } n = 0, 1, 2, \ldots$$

Notice at once that since the identity e belongs to G_n for each n, we have $P_n \subseteq P_{n+1}$. Now set

$$A_0 = \{e\},\ A_1 = P_1 \backslash P_0,\ \ldots,\ A_n = P_n \backslash P_{n-1}.$$

Each A_n is finite because each P_n is. Moreover, the A_n's are pairwise disjoint with $\bigcup_{n=0}^{\infty} A_n = G$.

Claim: For $g \in G_k$ and $l > k$, we have $gA_l \subseteq A_{l-1} \cup A_l \cup A_{l+1} = P_{l+1} \backslash P_{l-2}$.

Fix $g \in G_k$. Keeping in mind that $k < l$ and that the G_n's are nested, we know that $g \in G_s$ for $s \geq l-1$. Therefore, $gP_s \subseteq P_{s+1}$ for $s \geq l-2$. Take $p \in A_l \subseteq P_l$, from above we have that $gp \in P_{l+1}$. Now suppose that $gp \in P_{l-2}$ as well, then by above and the fact that the G_n's are closed under inverses, $p = g^{-1}(gp) \in P_{l-1}$ a contradiction because $p \in A_l = P_l \backslash P_{l-1}$. This proves the claim.

Since ω is a rare point, there exists a $U \in \mathcal{U}_\omega$ such that $|U \cap A_n| \leq 1$ for every n. Let

$$E = \bigcup_{n=0}^{\infty} A_{2n}, \quad \text{and} \quad E^c = O = \bigcup_{n=0}^{\infty} A_{2n+1}.$$

Then $\mathcal{U}_\omega$ being an ultrafilter, either $E \in \mathcal{U}_\omega$ or $O \in \mathcal{U}_\omega$. Assume the former, and set $V = U \cap E \in \mathcal{U}_\omega$.

Claim: For $g \in G_k$, $\text{card}(gV \cap V) \leq k/2$.

To see this write $V = \{a_{2m}\ ;\ a_{2m} \in A_{2m},\ \text{for some } m's\}$. If we take $x \in gV \cap V$, then $x = g \cdot a_{2m} = a_{2n}$ for some $m, n \in \mathbb{N}$. If $2m > k$, we know from our previous claim that $a_{2n} = g \cdot a_{2m} \in A_{2m-1} \cup A_{2m} \cup A_{2m+1}$, which forces $2n = 2m$ and in turn $a_{2n} = a_{2m}$, a contradiction. Therefore the only elements in $gV \cap V$ are for those m satisfying $2m \leq k$. That proves the claim.

Let $\mathcal{V} \subseteq \beta G$ be the open neighborhood of ω such that $\mathcal{V} \cap G = V$. This says that $g\mathcal{V} \cap \mathcal{V} \subseteq G$ and a finite set. Therefore, $\tilde{\mathcal{V}} = \mathcal{V} \cap G^*$ is our relatively open neighborhood such that $g\tilde{\mathcal{V}} \cap \tilde{\mathcal{V}} = \emptyset$ for all $g \neq e$. This condition ensures, by multiplying by suitable inverses, that $g_1\tilde{\mathcal{V}} \cap g_2\tilde{\mathcal{V}} = \emptyset$. $\square$

The following corollary is somewhat eye boggling when thinking in terms of topological dynamics.

Corollary 11.6. *Assuming the continuum hypothesis, if G is any countable discrete group, the set of wandering points in G^* is dense.*

Proof. Under the continuum hypothesis, rare points are dense in G^*. □

Notes:

(1) Consider the set of pairs

$$\mathcal{G} = \{(G, \phi)\ ;\ G \text{ is a countable discrete group and } \phi : \mathbb{N} \to G \text{ is a bijection } \}$$

For each pair we have a homeomorphic extension $\tilde{\phi} : \beta\mathbb{N} \to \beta G$. Notice that a rare ultrafilter $\omega \in \beta\mathbb{N}$ maps to a rare ultrafilter $\tilde{\phi}(\omega) \in \beta G$. Hence when $\omega \in \beta\mathbb{N}$ is rare, for every pair in $\mathcal{G}$, $\tilde{\phi}(\omega)$ is wandering in the Corona G^*. Is the converse statement true? More precisely, given $\omega \in \mathbb{N}$, if for every such pair (G, ϕ) $\tilde{\phi}(\omega)$ is wandering in G^*, can we say that ω is rare?

(2) We can also inquire as to whether the above Corollary holds without assuming the continuum hypothesis.

(3) Reid showed that for ω a rare ultrafilter, the corresponding state s_ω has a unique extension. Now we know that rare ultrafilters are always wandering. This motivates the following conjecture.

Conjecture: Let G be a countable discrete group and $\omega \in G^*$ wandering. If $\phi : \mathbb{N} \to G$ is a bijection, and $\tilde{\omega} = \phi^{-1}(\omega) \in \mathbb{N}^*$, does $s_{\tilde{\omega}}$ have a unique extension?

12. Groups Acting on Algebras

Definition 12.1. *Let $\mathcal{A}$ be an algebra and G a group. By an action of G on $\mathcal{A}$ we mean a homomorphism*

$$\alpha : G \longrightarrow Aut(\mathcal{A}).$$

Example: If a group G acts on a set X, G also acts on $\mathcal{F}(X) = \mathbb{C}^X$ as follows

$$\alpha : G \longrightarrow \mathrm{Aut}(\mathbb{C}^X)$$
$$[\alpha_g \cdot f](x) = f(g^{-1} \cdot x)$$

There are two things to check. First, it is not too difficult to verify that α_g is actually an automorphism. Second, α is a homomorphism. Indeed,

$$\begin{aligned}[\alpha_{gh}(f)](x) &= f((gh)^{-1} \cdot x) = f((h^{-1}g^{-1}) \cdot x) = f((h^{-1})g^{-1} \cdot x) \\ &= [\alpha_h \cdot f](g^{-1} \cdot x) = [(\alpha_h \circ \alpha_g) \cdot (f)](x),\end{aligned}$$

so that $\alpha_{gh} \cdot f = (\alpha_g \circ \alpha_h) \cdot f$ for all $f \in \mathbb{C}^X$, whence $\alpha_{gh} = \alpha_g \circ \alpha_h$.

Example: Now X is a topological space, and a group G acts by homeomorphisms of X, that is we have a homomorphism of groups

$$\begin{aligned} \sigma : G &\longrightarrow \text{Perm}(X) \\ G \ni g &\longmapsto \sigma_g \in \text{Perm}(X) \end{aligned}$$

where each σ_g is a homeomorphism of the space X. We the define

$$\alpha : G \longrightarrow \text{Aut}(C(X)), \quad g \longmapsto \alpha_g,$$

where $\alpha_g(f) = f \circ \sigma_{g^{-1}}$. The latter is clearly continuous, being the composition of continuous functions, and by same argument as in the above example, α is a homomorphism.

Example: Take $\mathcal{H}$ a Hilbert space, $\mathcal{A} \subseteq \mathcal{B}(\mathcal{H})$ a subalgebra, and $\{U_g\}_g$ a group of unitaries whereby $U_g \mathcal{A} U_g^{-1} \subseteq \mathcal{A}$ for all U_g's. Then the mapping $\mathcal{A} \to \mathcal{A}$ given by

$$a \longmapsto U_g a U_g^{-1}$$

is easily seen to be a homomorphism from the algebra $\mathcal{A}$ to itself for every fixed unitary U_g. Now since $U_g \mathcal{A} U_g^{-1} \subseteq \mathcal{A}$ holds for all g, replacing g by g^{-1} yields $U_g^{-1} \mathcal{A} U_g \subseteq \mathcal{A}$ as well. Together,

$$\mathcal{A} = U_g(U_g^{-1} \mathcal{A} U_g) U_g^{-1} \subseteq U_g \mathcal{A} U_g^{-1} \subseteq \mathcal{A}$$

so that $\mathcal{A} = U_g \mathcal{A} U_g^{-1}$ and the above map is actually an automorphism. Now look at

$$\alpha : G \longrightarrow \text{Aut}(\mathcal{A}) \ \text{ where } \ \alpha_g(a) = U_g a U_g^{-1}$$

Again, α is a homomorphism of groups. Indeed,

$$\begin{aligned} \alpha_{gh}(a) &= U_{gh} a U_{gh}^{-1} = U_{gh} a U_{(gh)^{-1}} = U_{gh} a U_{(h^{-1}g^{-1})} = U_g (U_h a U_{h^{-1}}) U_{g^{-1}} \\ &= U_g (U_h a U_h^{-1}) U_g^{-1} = (\alpha_g \circ \alpha_h)(a) \end{aligned}$$

thus $\alpha_{gh} = \alpha_g \circ \alpha_h$ as claimed.

Comment: Given $\alpha : G \to Aut(\mathcal{A})$, write $\alpha_g(a) = \alpha(g)(a)$ to avoid so many parathenses. We saw that for any $\mathcal{A} \subseteq B(\mathcal{H})$, $\{U_g\}$-group of unitaries, we get $U_g \mathcal{A} U_g^{-1} \subseteq \mathcal{A}, \forall g$. Then, the setting $\alpha_g(A) = U_g A U_{g^{-1}}$ defines an action of the group G on $\mathcal{A}$. Note that $U_g A = U_g A U_g^{-1} U_g = \alpha_g(A) \cdot U_g$. Let $\mathcal{B} = span\{AU_g : A \in \mathcal{A}, U_g \text{ unitary}\}$. Then,

$$(A_1 U_{g_1})(A_2 U_{g_2}) = A_1 U_{g_1} A_2 U_{g_1}^{-1} U_{g_1} U_{g_2} = A_1 \alpha_{g_1}(A_2) U_{g_1 g_2} \in \mathcal{B}$$

makes $\mathcal{B}$ an algebra. And, the general product

$$\Big(\sum_g A_g \cdot U_g\Big)\Big(\sum_h \tilde{A}_h \cdot U_h\Big) = \sum_{g,h} A_g \alpha_g(\tilde{A}_h) \cdot U_{gh}$$

motivates the following definition:

Definition 12.2. *Let $\mathcal{A}$ be an algebra, G a group, $\alpha : G \to Aut(\mathcal{A})$ group homomorphism. Then,*

$$\mathcal{A} \times_\alpha G = \Big\{ \sum_g^{finite} A_g \cdot g : A_g \in \mathcal{A}, g \in G \Big\}$$

is called **the crossed product algebra**, *with product defined as*

$$\Big(\sum_g A_g \cdot g\Big)\Big(\sum_h \tilde{A}_h \cdot h\Big) = \sum_{g,h} A_g \alpha_g(\tilde{A}_h) \cdot gh.$$

Written differently, let $gh = g'$, then $h = g^{-1}g'$ and

$$\Big(\sum_g A_g \cdot g\Big)\Big(\sum_h \tilde{A}_h \cdot h\Big) = \sum_{g,g'} A_g \alpha_g(\tilde{A}_{g^{-1}g'}) \cdot g'.$$

Function Viewpoint: The crossed product algebra $\mathcal{A} \times_\alpha G$ can be thought as the set of all finitely supported functions $f : G \to \mathcal{A}$ with the product given as

$$(f_1 * f_2)(g') = \sum_g f_1(g) \alpha_g(f_2(g^{-1}g')),$$

which is called, in fact, **"the twisted convolution product".**

Suppose, for a moment, $\mathcal{A} \subseteq B(\mathcal{H})$ is a C^*-algebra, then we have

$$(A \cdot U_g)^* = U_g^* A^* = U_{g^{-1}} A^* = \alpha_{g^{-1}}(A^*) U_{g^{-1}}, A \in \mathcal{A}.$$

This shows that when $\mathcal{A}$ is a C^*-algebra, then $\mathcal{B} = span\{AU_g : A \in \mathcal{A}, U_g \text{ unitary}\}$ is a $*$-algebra, with $B = \sum_g A_g U_g$, $B^* = \sum_g \alpha_{g^{-1}}(A^*) U_{g^{-1}}$. So, we can define a $*$-operation on the algebra $\mathcal{A} \times_\alpha G$, whenever $\mathcal{A}$ is a C^*-algebra, as $\left(\sum A_g \cdot g \right)^* = \sum \alpha_{g^{-1}}(A^*) \cdot g^{-1}$.

Function Viewpoint: $(f^*)(g) = \alpha_g(f(g^{-1})^*)$.

12.1. **Covariant Representations.** .

Let $\mathcal{A}$ be an algebra, $\alpha : G \to Aut(\mathcal{A})$ the action, denote this by $(\mathcal{A}, G, \alpha)$.

Definition 12.3. *A* **covariant representation** *of $(\mathcal{A}, G, \alpha)$ on a Hilbert space $\mathcal{H}$, is a pair (ρ, π) such that:*

1) *$\rho : \mathcal{A} \to B(\mathcal{H})$ is an algebra homomorphism.*
   Note: If $\mathcal{A}$ is unital, then $\rho(1) = I_{\mathcal{H}}$,
   and if $\mathcal{A}$ is a C^*-algebra, then ρ is a $*$-homomorphism.
2) *$\pi : G \to \mathcal{U}(\mathcal{H})$ is a group homomorphism.*
3) *$\pi(g)\rho(a)\pi(g^{-1}) = \rho(\alpha_g(a))$.*

Given a covariant representation (ρ, π), define a map $\rho \times_\alpha \pi : \mathcal{A} \times_\alpha G \to B(\mathcal{H})$,

$$(\rho \times_\alpha \pi)(\sum_g A_g \cdot g) = \sum_g \rho(A_g)\pi(g).$$

Since

$$\begin{aligned} \rho(A_g)\pi(g)\rho(A'_h)\pi(h) &= \rho(A_g)\pi(g)\rho(A'_h)\pi(g^{-1})\pi(gh) \\ &= \rho(A_g)\rho(\alpha_g(A'_h))\pi(gh) \\ &= \rho(A_g\alpha_g(A'_h)\pi(gh), \end{aligned}$$

then $(\rho \times_\alpha \pi)(A_g \cdot g)(\rho \times_\alpha \pi)(A'_h \cdot h) = (\rho \times_\alpha \pi)\big((A_g \cdot g)(A'_h \cdot h)\big)$.
This shows that any covariant representation (ρ, π) gives rise to a representation $\rho \times_\alpha \pi$ of $\mathcal{A} \times_\alpha G$.
Conversely, if given a homomorphism $\tilde{\pi} : \mathcal{A} \times_\alpha G \to B(\mathcal{H})$, where $\mathcal{A}$ is unital, and set $\rho(A) = \tilde{\pi}(A \cdot e)$, $e \in G$ is the identity, and $\pi(g) = \tilde{\pi}(1_{\mathcal{A}} \cdot g)$, then (ρ, π) is a covariant representation.

Example 15 Let G be a countable discrete group, $\mathcal{H} = l^2(G)$, let $C(\beta G) \cong l^\infty(G)$, identify $l^\infty(G) \cong \mathcal{D} \subseteq B(l^2(G))$, where given $f \in l^\infty(G)$, write D_f for the bounded diagonal operator, such that $D_f \cdot e_h = f(h) \cdot e_h$. So, we really have $\rho : l^\infty(G) \to B(l^2(G))$ given by $\rho(f) = D_f$, and we have $\pi = \lambda : G \to B(l^2(G))$, where $\lambda(g) = U_g$, $U_g \cdot e_h = e_{gh}$.
Now, compute $\pi(g)\rho(f)\pi(g^{-1}) = U_g D_f U_{g^{-1}}$:

$$\begin{aligned} \langle U_g D_f U_{g^{-1}} e_{h_1}, e_{h_2}\rangle &= \langle U_g D_f e_{g^{-1}h_1}, e_{h_2}\rangle \\ &= \langle U_g f(g^{-1}h_1) e_{g^{-1}h_1}, e_{h_2}\rangle \\ &= f(g^{-1}h_1)\langle e_{h_1}, e_{h_2}\rangle \\ &= \left\{ \begin{array}{ll} f(g^{-1}h_1), & h_1 = h_2 \\ 0, & h_1 \neq h_2 \end{array} \right. . \end{aligned}$$

Therefore, $U_g D_f U_{g^{-1}} = D_{f_1}$, where $f_1(h_1) = f(g^{-1}h_1) = \alpha_g(f)(h_1)$, which implies $U_g D_f U_{g^{-1}} = D_{\alpha_g(f)}$. Hence, $\pi(g)\rho(f)\pi(g^{-1}) = \rho(\alpha_g(f)) \Rightarrow (\rho, \pi)$ is a covariant representation for the action of G on βG.

Example 16 Let X be a topological space, G a group with action on X, pick $x_e \in X$. Look at the homomorphism $\rho : C(X) \to l^\infty(G) \subseteq B(l^2(G))$ given by $\rho(f)(g) = f(g \cdot x_e)$. Let $\pi = \lambda : G \to B(l^2(G))$.

Then $\pi(g)\rho(f)\pi(g^{-1}) = U_g\rho(f)U_g^{-1}$.

$$\begin{aligned}\langle U_g\rho(f)U_g^{-1}e_{h_1}, e_{h_2}\rangle &= \langle U_g D_f e_{g^{-1}h_1}, e_{h_2}\rangle \\ &= \langle U_g f((g^{-1}h_1)\cdot x_e)e_{g^{-1}h_1}, e_{h_2}\rangle \\ &= f((g^{-1}h_1)\cdot x_e)\langle e_{h_1}, e_{h_2}\rangle \\ &= \begin{cases} f((g^{-1}h_1)\cdot x_e), & h_1 = h_2 \\ 0, & h_1 \neq h_2 \end{cases}.\end{aligned}$$

Therefore, $U_g\rho(f)U_g^{-1} = \rho(\alpha_g(f))$.
(You can double check by doing inner-products with vectors.)

13. DYNAMICAL SYSTEMS AND KADISON SINGER

Let G be a countable, discrete group. By defining any 1-1, onto map between G and $\mathbb{N}$. We may identify $l^2(G) = l^2(\mathbb{N})$, $l^\infty(G) = l^\infty(\mathbb{N})$ and when we look at $B(l^2(G))$, the diagonal operators with respect to the orthonormal basis, $\{e_g : g \in G\}$ are identified with $l^\infty(G)$.
So we may regard $l^\infty(G) \cong \mathcal{D} \subseteq B(l^2(G))$ as another model for our discrete MASA.
Fix w $\in \beta G$, define $s_w : l^\infty(G) \to \mathbb{C}$ via $s_w(f) = f(w)$ by identifying $l^\infty(G) = C(\beta G)$.
Let s : B($l^2(G)$) $\to \mathbb{C}$ be any state that extends s_w.
By GNS representation of s, there exists $\mathcal{H}$ Hilbert space, $v_e \in \mathcal{H}$ and $\pi : B(l^2(G)) \to B(\mathcal{H})$ such that s(X)=$<\pi(X)v_e, v_e>$.

Let $v_g = \pi(U_g)v_e$ and let $\mathcal{L}_s$=span $\overline{\{v_g : g \in G\}}^{\|\cdot\|}$.
Define $\varphi_s : B(l^2(G)) \to B(\mathcal{L}_s)$ by $\varphi_s(X) = P_{\mathcal{L}_s}\pi(X)|_{\mathcal{L}_s}$ -this is a completely positive map which is to be defined later.

Let $f \in C(\beta G) = l^\infty(G)$, let D_f be the corresponding diagonal operator, $D_f e_h = f(h)e_h$.
Look at

$$< \pi(D_f)v_g, v_g >=< \pi(D_f)\pi(U_g)v_e, \pi(U_g)v_e >=< \pi(U_g^{-1}D_fU_g)v_e, v_e >$$

Recall $U_gD_fU_g^{-1} = D_f$ where $f\ (h) = f(g^{-1}h)$.
$\Rightarrow U_g^{-1}D_fU_g = D_{\hat{f}}$ where $\hat{f}(h) = f(gh)$.

$$\therefore < \pi(U_g^{-1}D_fU_g)v_e, v_e >=< \pi(D_{\hat{f}})v_e, v_e >= s_w(D_{\hat{f}}) = \hat{f}(w) = f(g.w)$$

Therefore the map $\pi(D_f) \mapsto < \pi(D_f)v_g, v_g >= f(g.w)$ is a homomorphism.
$\Rightarrow v_g$ is an eigenvector for $\pi(D_f)$ with eigenvalue f(g.w).
$\Rightarrow \pi(D_f)$ is the diagonal operator with respect to the basis vector $\{v_g : g \in G\}$.
Each v_g is an eigenvector of the normal operator $\pi(D_f)$, $v_g \perp v_h, g \neq h$.

Define a map $W : l^2(G) \to \mathcal{L}_s$ by W$(e_h) = v_h$ which extends to a unitary.

$$\begin{aligned} W^{-1}\pi(U_g)We_h &= W^{-1}\pi(U_g)v_h \\ &= W^{-1}\pi(U_g)\pi(U_h)v_e \\ &= W^{-1}\pi(gh)v_e \\ &= W^{-1}v_{gh} \\ &= e_{gh} = U_g e_h. \end{aligned}$$

$\therefore W^{-1}\pi(U_g)W = U_g$.

$$\begin{aligned} W^{-1}\pi(D_f)We_h &= W^{-1}\pi(D_f)v_h \\ &= W^{-1}f(h.w)v_h \\ &= f(h.w)e_h. \end{aligned}$$

$\therefore W^{-1}\pi(D_f)W = D_{\hat{f}}$ where $\hat{f}(g) = f(g.w)$.
Let $\phi_s : B(l^2(G)) \to B(l^2(G))$ be defined by $\phi_s(X) = W^{-1}\varphi_s(X)W$.
Note that ϕ_s is a completely positive map with the action on unitaries and diagonals as $\phi_s(U_g) = U_g$, $\phi_s(D_f) = D_{\hat{f}}$ where $\hat{f}(g) = f(g.w)$.

$$\begin{aligned} < \phi_s(X)e_h, e_g > &=< W^{-1}\varphi_s(X)We_h, e_g > \\ &=< \varphi_s(X)We_h, We_g > \\ &=< \varphi_s(X)v_h, v_g > \\ &=< P_{\mathcal{L}_s}\pi(X)|_{\mathcal{L}_s}v_h, v_g > \\ &=< \pi(X)v_h, v_g > \\ &=< \pi(X)\pi(U_h)v_e, \pi(U_g)v_e > \\ &=< \pi(U_g^{-1}XU_h)v_e, v_e > \\ &= s(U_g^{-1}XU_h) \end{aligned}$$

In summary, given s- state extension of s_w, we get a map $\phi_s : B(l^2(G)) \to B(l^2(G))$ such that X $\mapsto \phi_s(X) = (s(U_g^{-1}XU_h))_{g,h}$ is a completely positive map and $\phi_s(U_g) = U_g$, $\phi_s(D_f) = D_{\hat{f}}$ where $\hat{f}(g) = f(g.w)$.
Define $\pi_w : \mathcal{D} \to \mathcal{D}$, $\pi_w(D_f) = D_{\hat{f}}$.
When we identify $\mathcal{D} = C(\beta G)$, then $\pi_w(f) = \hat{f}$ where $\hat{f}(g) = f(g.w)$.

$$\pi_w(f_1f_2)(g) = \widehat{f_1f_2}(g) = \hat{f}_1(g)\hat{f}_2(g) = \pi_w(f_1)(g)\pi_w(f_2)(g)$$

$\therefore \pi_w$ is a homomorphism on $\mathcal{D}$.

$$\begin{aligned} < \phi_s(D_f U_g) e_{h_1}, e_{h_2} > &= s(U_{h_2}^{-1} D_f U_g U_{h_1}) \\ &= s(U_{h_2}^{-1} D_f U_{gh_1}) \\ &= < \phi_s(D_f) e_{gh_1}, e_{h_2} > \\ &= < \phi_s(D_f) U_g e_{h_1}, e_{h_2} > \end{aligned}$$

$\therefore \phi_s(D_f U_g) = \phi_s(D_f) U_g = \pi_w(D_f) U_g$.
Similarly, $\phi_s(U_g D_f) = U_g \phi_s(D_f) = U_g \pi_w(D_f)$.

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