QIC890/PMATH950
HOMEWORK SET 1
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1. Problem

Let X and Y be Banach spaces, let $V \subseteq X$ be a vector subspace and let W denote its closure. Given $T : V \to Y$ a bounded linear map and $w \in W$, show that if $\{v_n\} \subseteq V$ is any sequence such that $\|w - v_n\| \to 0$, then there exists a vector $y \in Y$ such that $\|y - Tv_n\| \to 0$. Show that y is independent of the particular sequence $\{v_n\}$ converging to w and that setting $R : W \to Y$ by $Rw = y$ where y is the unique vector obtained in this manner defines a bounded linear map that extends T with $\|R\| = \|T\|$. We will refer to this process as **extension by continuity** and refer to R as the closure of T.

2. Problem

Let $\mathcal{H}$ be a separable Hilbert space with a countably infinite o.n.b. given by $\{e_n : n \in \mathbb{N}\}$. Let $\lambda_n \in \mathbb{C}$ with $\lambda_n \neq 0$ for every n. Set $T(\sum_n a_n e_n) = \sum_n a_n \lambda_n e_{n+1}$. Prove:

(1) $T \in B(\mathcal{H}) \iff \sup\{|\lambda_n| : n \in \mathbb{N}\} < +\infty$ and that in this case $\|T\| = \sup\{|\lambda_n| : n \in \mathbb{N}\}$
(2) $T \in \mathbb{K}(\mathcal{H}) \iff \lim_n \lambda_n = 0$.
(3) $T \in \mathcal{C}_p(\mathcal{H}) \iff \sum_n |\lambda_n|^p < +\infty$ and that $\|T\|_p = \big(\sum_n |\lambda_n|^p\big)^{1/p}$.
(4) In the polar decomposition of $T = W|T|$, with $W : \mathcal{R}(|T|)^- \to \mathcal{R}(T)^-$ show that the unitary W can never extended to a unitary on $\mathcal{H}$.

3. Problem

Let $\mathcal{H}$ be a Hilbert space with a possibly uncountable o.n.b. $\{\phi_a : a \in A\}$. Prove that for each vector $h \in \mathcal{H}$ there is at most a countable set of $a \in A$ such that $\langle \phi_a | h \rangle \neq 0$.

4. Problem

Let T be an $n \times n$ matrix, which we think of as an element of $B(\mathbb{C}^n)$, where $\mathbb{C}^n$ is the standard n dimensional Hilbert space with o.n.b. $\{e_1, ..., e_n\}$. Prove that in this case we may write $T = U|T|$ where $U : \mathbb{C}^n \to \mathbb{C}^n$ is a unitary. Show that there are two unitary matrices U_1 and U_2 such that

$T = U_1 D U_2$ where D is the diagonal matrix whose entries are the singular numbers of T.

5. Problem

Let $\mathcal{H}$ be a Hilbert space with a countable o.n.b. $\{e_n : n \in \mathbb{N}\}$. Given $T \in B(\mathcal{H})$ we define the **matrix of T with respect to the o.n.b.** to be the array $(t_{i,j})_{i,j\in\mathbb{N}}$ where

$$t_{i,j} = \langle e_i | T e_j \rangle.$$

Prove that:

(1) for each $i, \sum_j |t_{i,j}|^2 < +\infty$,
(2) for each $j, \sum_i |t_{i,j}|^2 < +\infty$,
(3) if $h \in \mathcal{H}$ with $h = \sum_j a_j e_j$ then $Th = \sum_{i,j} t_{i,j} a_j e_i$,
(4) $T \in \mathcal{C}_2(\mathcal{H}) \iff \sum_{i,j} |t_{i,j}|^2 < +\infty$ and that in this case $\|T\|_2 = \big(\sum_{i,j} |t_{i,j}|^2\big)^{1/2}$.

Finally, give an example of a $p, 1 < p < +\infty$ and $T \in \mathcal{C}_p(\mathcal{H})$ such that $\sum_{i,j} |t_{i,j}|^p = +\infty$.

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