1. Problem

Let $X$ and $Y$ be Banach spaces, let $V \subseteq X$ be a vector subspace and let $W$ denote its closure. Given $T : V \to Y$ a bounded linear map and $w \in W$, show that if $\{v_n\} \subseteq V$ is any sequence such that $\|w - v_n\| \to 0$, then there exists a vector $y \in Y$ such that $\|y - Tv_n\| \to 0$. Show that $y$ is independent of the particular sequence $\{v_n\}$ converging to $w$ and that setting $R : W \to Y$ by $Rw = y$ where $y$ is the unique vector obtained in this manner defines a bounded linear map that extends $T$ with $\|R\| = \|T\|$. We will refer to this process as extension by continuity and refer to $R$ as the closure of $T$.

2. Problem

Let $H$ be a separable Hilbert space with a countably infinite o.n.b. given by $\{e_n : n \in \mathbb{N}\}$. Let $\lambda_n \in \mathbb{C}$ with $\lambda_n \neq 0$ for every $n$. Set $T(\sum_n a_n e_n) = \sum_n a_n \lambda_n e_{n+1}$. Prove:

(1) $T \in B(H) \iff \sup\{\|\lambda_n\| : n \in \mathbb{N}\} < +\infty$ and that in this case $\|T\| = \sup\{\|\lambda_n\| : n \in \mathbb{N}\}$

(2) $T \in K(H) \iff \lim_n \lambda_n = 0$.

(3) $T \in C_p(H) \iff \sum_n |\lambda_n|^p < +\infty$ and that $\|T\|_p = \left(\sum_n |\lambda_n|^p\right)^{1/p}$.

(4) In the polar decomposition of $T = W|T|$, with $W : \mathcal{R}(|T|)^{-} \to \mathcal{R}(T)^{-}$ show that the unitary $W$ can never extended to a unitary on $H$.

3. Problem

Let $\mathcal{H}$ be a separable Hilbert space with a possibly uncountable o.n.b. $\{\phi_a : a \in A\}$. Prove that for each vector $h \in \mathcal{H}$ there is at most a countable set of $a \in A$ such that $\langle \phi_a | h \rangle \neq 0$.

4. Problem

Let $T$ be an $n \times n$ matrix, which we think of as an element of $B(\mathbb{C}^n)$, where $\mathbb{C}^n$ is the standard $n$ dimensional Hilbert space with o.n.b. $\{e_1, ..., e_n\}$. Prove that in this case we may write $T = U|T|$ where $U : \mathbb{C}^n \to \mathbb{C}^n$ is a unitary. Show that there are two unitary matrices $U_1$ and $U_2$ such that
\[ T = U_1DU_2 \] where \( D \) is the diagonal matrix whose entries are the singular numbers of \( T \).

5. Problem

Let \( \mathcal{H} \) be a Hilbert space with a countable o.n.b. \( \{ e_n : n \in \mathbb{N} \} \). Given \( T \in B(\mathcal{H}) \) we define the **matrix of \( T \) with respect to the o.n.b.** to be the array \((t_{i,j})_{i,j \in \mathbb{N}}\) where

\[ t_{i,j} = \langle e_i | Te_j \rangle. \]

Prove that:

1. for each \( i \), \( \sum_j |t_{i,j}|^2 < +\infty \),
2. for each \( j \), \( \sum_i |t_{i,j}|^2 < +\infty \),
3. if \( h \in \mathcal{H} \) with \( h = \sum_j a_je_j \) then \( Th = \sum_{i,j} t_{i,j} a_je_i \),
4. \( T \in \mathcal{C}_2(\mathcal{H}) \iff \sum_{i,j} |t_{i,j}|^2 < +\infty \) and that in this case \( \|T\|_2 = \left( \sum_{i,j} |t_{i,j}|^2 \right)^{1/2} \).

Finally, give an example of a \( p, 1 < p < +\infty \) and \( T \in \mathcal{C}_p(\mathcal{H}) \) such that \( \sum_{i,j} |t_{i,j}|^p = +\infty \).