Corollary: Let ϕ be simple and $\phi = \sum_{l=1}^{m} b_l \chi_{F_l}$, may not be in standard form, but each $b_l \geq 0$. Then, $\int \phi d\mu = \sum_{l=1}^{m} b_l \mu(F_l)$.

Proof: By Proposition 2.14-(b) and induction, $\int \phi d\mu = \sum_{l=1}^{m} [\int b_l \chi_{F_l} d\mu]$

$$= \sum_{l=1}^{m} \int (b_l \chi_{F_l} + 0 \cdot \chi_{F_l^c}) d\mu \text{ (notice that this is in standard form)}$$

$$= \sum_{l=1}^{m} b_l \mu(F_l) + 0 \cdot \mu(F_l^c) = \sum_{l=1}^{m} b_l \mu(F_l).$$

Definition: Let $(X, \mathcal{M}, \mu)$ be a measure space. For, $f \in L^+$, we define $\int f d\mu = sup\{\int \phi d\mu : 0 \leq \phi \leq f$ and ϕ is simple.$\}$

Prop'n ~~Note~~ (1): Let ψ be simple. Then. $\int \psi d\mu = sup\{\int \phi d\mu : 0 \leq \phi \leq \psi$ and ϕ is simple. We have no trouble here, sup gives the same value for a simple function ψ.

Note (2): If $f \in L^+$ and $c > 0$, then $\int cf d\mu = sup\{\int \phi d\mu : 0 \leq \phi \leq cf$ and ϕ is simple.$\}$ Let $\psi = c^{-1}\phi$, that is $c\psi = \phi$, where ψ is simple. Then, $\phi \leq cf$ and ϕ is simple $\Leftrightarrow c\psi \leq cf$ and ψ is simple $\Leftrightarrow \psi \leq f$ and ψ is simple. So, $\int cf d\mu = sup\{\int \phi d\mu : 0 \leq \phi \leq cf$ and ϕ is simple.$\}$ =

$sup\{\int \phi d\mu : 0 \leq \psi \leq f, \psi$ is simple and ψ=$c^{-1}\phi$.$\}$ =

$sup\{\int c\psi d\mu : 0 \leq \psi \leq f$, and ψ is simple.$\}$ =

$csup\{\int \psi d\mu : 0 \leq \psi \leq f$, and ψ is simple.$\}$ =

$c\int f d\mu$.

Prop'n ~~Note (3)~~: If $f, g \in L^+$ and $f \leq g$, then $\int f d\mu \leq \int g d\mu$ because there are more ϕ's in the definition of sup for $\int g d\mu$.

ENDED HERE >

Theorem (The Monotone Convergence Theorem): Suppose that $\{f_n\} \subseteq L^+$ and $f_j \leq f_{j+1}$ for all j. Let $f(x) = \sup_j f_j(x) = \lim_j f(x)$.

Then, $\int f d\mu = \lim_j \int f_j d\mu = \sup_j \int f_j d\mu$.

Proof: [Show that $\sup_j \int f_j d\mu \leq \int f d\mu$.]

Since each $f_j \leq f$, $\int f_j d\mu \leq \int f d\mu \Rightarrow \sup_j \int f_j d\mu \leq \int f d\mu$.

[Show that $\int f d\mu \le \sup_j \int f_j d\mu$.]

Fix α where $0 < \alpha < 1$. Take $0 \le \phi \le f$ where ϕ is simple. Then, $\alpha\phi < f$. Let $E_n = \{x : f_n(x) \ge \alpha\phi(x)\}$. Then, each $E_n \in \mathcal{M}$, $E_1 \subseteq E_2 \subseteq \cdots$ because f_n is increasing, and $\bigcup_j E_j = X$. Now, $\int f_n d\mu \ge \int_{E_n} f_n d\mu = \int f_n \chi_{E_n} d\mu \ge \int_{E_n} \alpha\phi d\mu$ $= \alpha\int_{E_n} \phi d\mu$. Thus, $\sup_n \int f_n d\mu \ge \alpha \int_{E_n} \phi d\mu$ for all n.

By Proposition 2.13-(d), $\int_{E_n} \alpha\phi d\mu = \nu(E_n)$ is a measure which is continuous from below. $\Rightarrow \nu(X) = \lim_n \nu(E_n) \Rightarrow \int_X \alpha\phi d\mu =$ $\int \alpha\phi d\mu = \lim_n \int_{E_n} \alpha\phi d\mu \Rightarrow \sup_n \int f_n d\mu \ge \alpha \int \phi d\mu$ which is true for all $\phi \le f \Rightarrow \sup_n \int f_n d\mu \ge \alpha \cdot \sup\{\int \phi d\mu : \phi \le f$ and ϕ is simple$\} = \alpha \int f d\mu$ which is true for all $\alpha < 1 \Rightarrow \sup_n \int f_n d\mu$ $\ge \int f d\mu$.

Therefore, $\int f d\mu = \sup_n \int f_n d\mu$.

(1)

Corollary: Let $f \in L^+, 0 \le \phi_1 \le \phi_2 \le \cdots \le f$, each ϕ_n be simple, and $\lim_n \phi_n(x) = f(x)$, then $\int f d\mu = \lim_n \int \phi_n d\mu$.

(2) $f, g \in L^+$ then $\int (f+g) d\mu = \int f d\mu + \int g d\mu$

<u>Notation</u>: $f_1 \le f_2 \le \cdots \le f$ and $\lim_n f_n = f \Leftrightarrow f_n \nearrow f$

Example: Let $\mu(E) = 0$ and $f(x) = \begin{cases} +\infty & \text{if } x \in E \\ 0 & \text{if } x \notin E \end{cases}$

Also, let $\phi_n(x) = n \cdot \chi_E$. Then, $\phi_n \nearrow f$, and $\int f d\mu = \lim_n \int \phi_n d\mu =$ $\lim_n n\mu(E) = n \cdot 0 = 0$.

Theorem 2.15: Suppose that $\{f_n\} \subseteq L^+$, and $f(x) = \sum_{n=1}^{\infty} f_n(x)$. Then, $\int f d\mu = \sum_{n=1}^{\infty} \int f_n d\mu$.

Proof: ~~Let $\psi_j \nearrow f_1$ be simple, and also $\phi_i \nearrow f_2$ be simple, then $(\psi_j + \phi_i) \nearrow (f_1 + f_2)$, and $\int (f_1 + f_2) d\mu = \lim_n \int (\psi_n + \phi_n) d\mu$ $= \lim_n (\int \psi_n d\mu + \int \phi_n d\mu) = \lim_n \int \psi_n d\mu + \lim_n \int \phi_n d\mu = \int f_1 d\mu$ $+ \int f_2 d\mu$. Thus, $\int (f_1 + f_2) d\mu = \int f_1 d\mu + \int f_2 d\mu$.~~

$S_N = \sum_{n=1}^{N} f_n$, $S_N \nearrow f \Rightarrow \int f d\mu = \lim_N \int S_N d\mu = \lim_N \sum_{j=1}^{N} \int f_j d\mu$ $= \sum_{j=1}^{\infty} \int f_j d\mu$

Next, use induction to get $\int (f_1 + f_2 + \cdots + f_n)d\mu = \int f_1 d\mu + \int f_2 d\mu + \cdots + \int f_n d\mu$.
Let $S_N(x) = \sum_{n=1}^{N} f_n(x)$, then $S_N(x) \leq S_{n+1}(x) \leq \cdots$. So, $S_N \nearrow f$. Now $\int f d\mu = \lim_N \int S_N d\mu = \lim_N \int \sum_{n=1}^{N} f_n d\mu = \lim_N \sum_{n=1}^{N} \int f_n d\mu = \sum_{n=1}^{\infty} \int f_n d\mu$.

Proposition 2.16: If $f \in L^+$, then $\int f d\mu = 0 \Leftrightarrow f = 0$ *a.e.*

Proof: ($\Rightarrow$) suppose that $f \in L^+$ and $\int f d\mu = 0$. Let $0 \leq \phi \leq f$ and ϕ be simple. Then $0 \leq \int \phi d\mu \leq \int f d\mu \Rightarrow \int \phi d\mu = 0$. But, if we write $\phi = \sum_{j=1}^{n} a_j \chi_{E_j}$, then $0 = \int \phi d\mu = \sum_{j=1}^{n} a_j \mu(E_j)$ and $a_j \geq 0$. Thus, $a_j \neq 0 \Rightarrow \mu(E_j) = 0 \Rightarrow \phi = 0$ *a.e.* Now, let $0 \leq \phi_1 \leq \phi_2 \leq \cdots \leq f$, $\phi_n \nearrow f$ and ϕ_n be simple. Then, for each n, $\int \phi_n d\mu = 0 \Rightarrow \phi_n = 0$ *a.e.* Now, let $N_n = \{x : \phi_n(x) \neq 0\}$. Then, $\mu(N_n) = 0$. Let $N = \bigcup_{n=1}^{\infty} N_n$, then $\mu(N) = 0$. For $x \notin N$, $\phi_n(x) = 0$ for all n. This implies that $f(x) = \lim_n \phi_n(x) = 0$ if $x \notin N \Rightarrow \{x : f(x) \neq 0\} \subseteq N$. Thus, $\mu(\{x : f(x) \neq 0\}) \leq \mu(N) = 0$, and so $f = 0$ *a.e.*
($\Leftarrow$) Let $f(x) = 0$ *a.e.* Also, let $0 \leq \phi_1 \leq \phi_2 \leq \cdots \leq f$, $\phi_n \nearrow f$ and ϕ_n be simple. Then, $\phi_n = 0$ *a.e.* So, if $\phi_n = \sum_j a_j \chi_{E_j}$, then $\int \phi_n d\mu = \sum_j a_j \mu(E_j) = 0$ because if $a_j \neq 0$, then $\mu(E_j) = 0$. Thus, $\int f d\mu = \lim_n \int \phi_n d\mu = 0$.

Corollary 2.17: Let $\{f_n\} \subseteq L^+$, and $f_1(x) \leq f_2(x) \leq \cdots \leq f(x)$ for all $x \notin N$ where $\mu(N) = 0$. If $\lim_n f_n(x) = f(x)$, $x \notin N$, then $\int f d\mu = \lim_n \int f_n d\mu$.

Proof: Write $f = f \cdot \chi_N + f \cdot \chi_{N^c}$. Then, $\int f d\mu = \int f \cdot \chi_N d\mu + \int f \cdot \chi_{N^c} d\mu = \int f \cdot \chi_{N^c} d\mu$ because $\int f \cdot \chi_N d\mu = 0$. Thus, the Monotone Convergence Theorem, $\int f \cdot \chi_{N^c} d\mu = \lim_n \int f_n \chi_{N^c} d\mu$ since $f_n \chi_{N^c} \nearrow f \cdot \chi_{N^c}$. But, $\int f_n d\mu = \int f_n \chi_{N^c} d\mu + \int f_n \chi_N d\mu = \int f_n \chi_{N^c} d\mu$ because $\int f_n \chi_N d\mu = 0$ *a.e.* Thus, $\int f d\mu = \lim_n \int f_n d\mu$.

Notation: $f_n \nearrow f$ *a.e.* when the hypotheses of Corollary 2.17 are met [throw away where the measure is 0.]

Corollary 2.17 restated: If $f_n \nearrow f$ *a.e.*, $\int f d\mu = \lim_n \int f_n d\mu$.

Example: Importance of the hypothesis that $\{f_n\}$ be increasing.
Let $f_n = n \cdot \chi_{(0,1/n)}$. Then, $\lim_n f_n(x) = 0$. But, $\int f_n dm = n \cdot m((0,1/n))$ $= n \cdot (1/n - 0) = 1$. Thus, $\int f dm \neq \lim_n \int f_n dm$.
Notice that if $x \in (1/(n+1), 1/n)$, then $n = f_n(x) \geq f_{n+1}(x) = 0$. So, f_n is not increasing.

Fatou's Lemma 2.18: Let $\{f_n\} \subseteq L^+$. Let $f(x) = \liminf_n f_n(x)$ which is measurable (by Proposition 2.7.) Then, $\int f d\mu \leq \liminf_n \int f_n d\mu$.
[Note that $\liminf_n a_n = \lim_k b_k = \sup b_k$ where $b_k = \inf_{n \geq k} a_n$, $b_1 \leq b_2 \leq \cdots$]

Proof: Define $g_k(x) = \inf_{n \geq k} f_n(x)$. Then, g_k is measurable, $g_1 \leq g_2 \leq \cdots$, and $\lim_k g_k(x) = \liminf_n f_n(x)$. This implies that $g_k \nearrow f$. Note that $g_k \leq f_k$. Thus, $\int f d\mu = \lim_k \int g_k d\mu = \liminf_k \int g_k d\mu$ $\leq \liminf_k \int f_k d\mu$.

OMIT **Corollary 2.19**: Let $\{f_n\} \subseteq L^+$ and $f \in L^+$, and $f_n(x) \to f(x)$ *a.e.* Then, $\int f d\mu \leq \liminf_n \int f_n d\mu$.

Proposition 2.20: Let $f \in L^+$ and $\int f d\mu < +\infty$. Then, $\mu(\{x : f(x) = +\infty\}) = 0$ and $\{x : f(x) > 0\}$ is a σ-finite set.

Proof: Let $A = \{x : f(x) = +\infty\}$, and let $\phi_n = n\chi_A$. Then, $\phi_n \leq f$. This implies that $n\mu(A) = \int \phi_n d\mu \leq \int f d\mu < +\infty$ for all n $\Rightarrow \mu(A) = 0$ since the L.H.S. becomes unbounded when $\mu(A) \neq 0$. Let $E_n = \{x : f(x) \geq 1/n\}$. Then, $[x : f(x) \neq 0\} = \{x : f(x) > 0\} = \bigcup_{n=1}^{\infty} E_n$. Let $\phi_n = 1/n \cdot \chi_{E_n} \leq f$. Then, $1/n \cdot \mu(E_n) = \int \phi_n d\mu \leq \int f d\mu < +\infty \Rightarrow \mu(E_n) < +\infty$ for all $n \Rightarrow \{x : f(x) > 0\}$ is a σ-finite set.

2.3 Integration of Real and Complex Functions

Real Case: Let $(X, \mathcal{M}, \mu)$ be a measure space, $f : X \to \overline{\mathbb{R}}$ be measurable and write $f = f^+ - f^-$. Then, $|f| = f^+ + f^-$. So, $\int |f| d\mu = \int f^+ d\mu + \int f^- d\mu$, and $\int |f| d\mu < +\infty \Leftrightarrow \int f^+ d\mu < +\infty$ and $\int f^- d\mu < +\infty$.
If one of $\int f^+ d\mu$ and $\int f^- d\mu$ is finite, then we define $\int f d\mu = \int f^+ d\mu - \int f^- d\mu$.

Definition: If $f : X \to \overline{\mathbb{R}}$ is measurable and $\int |f| d\mu < +\infty$, then we say that f is **integrable**, and we let $\mathcal{L}^1(\mu) = \{f : X \to \overline{\mathbb{R}} : f$ is measurable and $\int |f| d\mu < +\infty\}$.

Complex Case: Let $f : X \to \mathbb{C}$ $(\mathbb{R} \times \mathbb{R})$, which is still Borel. Write $f = Ref + iImf$ where $Ref = Ref^+ - Ref^-$, and $Imf = Imf^+ - Imf^-$.
For complex numbers $z = a + ib$, $|z| = \sqrt{a^2 + b^2} \leq |a| + |b| \leq 2|z|$.
$|Ref| = Ref^+ + Ref^-$, $|Imf| = Imf^+ + Imf^-$ and $|f| \leq Ref^+ + Ref^- + Imf^+ + Imf^- \leq 2|f|$. So, $\int |f| d\mu < +\infty \Leftrightarrow \int Ref^+ d\mu < +\infty$, $\int Ref^- d\mu < +\infty$, $\int Imf^+ d\mu < +\infty$, and $\int Imf^- d\mu < +\infty$.

Definition: If $f : X \to \mathbb{C}$ is measurable and $\int |f| d\mu < +\infty$, then f is said to be **integrable**. $\mathcal{L}^1_{\mathbb{C}} = \{f : X \to \mathbb{C} : f$ is measurable and $\int |f| d\mu < +\infty\}$

Proposition 2.21: The set of integrable functions is a vector space. Also, if f and g are integrable and λ is a scalar, then $\int (f + g) d\mu = \int f d\mu + \int g d\mu$ and $\int \lambda f d\mu = \lambda \int f d\mu$.

Proof (Real case): [Show that $f + g$ and λf are integrable.]

Let f and g be integrable. Then, $\int |f| d\mu < +\infty$ and $\int |g| d\mu < +\infty$. First, $|f + g| \leq |f| + |g|$ implies that $\int |f + g| d\mu \leq \int (|f| + |g|) d\mu \leq \int |f| d\mu + \int |g| d\mu < +\infty$. Thus, $f + g$ is integrable. Next, $|\lambda f| = |\lambda||f|$. Thus, $\int |\lambda f| d\mu = \int |\lambda||f| d\mu = |\lambda| \int |f| d\mu < +\infty$, and so λf is also integrable. Hence, the set of integrable functions is a vector space.

[Show that $\int (f + g) d\mu = \int f d\mu + \int g d\mu$.]

Let $h = f + g$, and write $h = h^+ - h^-$. Then, $f + g = f^+ - f^- + g^+ - g^- \Rightarrow h^+ - h^- = f^+ - f^- + g^+ - g^- \Rightarrow h^+ + f^- + g^- = f^+ + g^+ + h^-$. Since f, g and h are all integrable, the integrals of all these 6 functions are finite.

Now, $\int(h^+ + f^- + g^-)d\mu = \int(f^+ + g^+ + h^-)d\mu$. Also, $\int(h^+ + f^- + g^-)d\mu = \int h^+ d\mu + \int f^- d\mu + \int g^- d\mu$, and $\int(f^+ + g^+ + h^-)d\mu = \int f^+ d\mu + \int g^+ d\mu + \int h^- d\mu$ by Theorem 2.15. This implies that $\int h^+ d\mu + \int f^- d\mu + \int g^- d\mu$ $= \int f^+ d\mu + \int g^+ d\mu + \int h^- d\mu \Rightarrow \int h^+ d\mu - \int h^- d\mu =$ $\int f^+ d\mu - \int f^- d\mu + \int g^+ d\mu - \int g^- d\mu \Rightarrow \int(f+g)d\mu =$ $\int h d\mu = \int f d\mu + \int g d\mu$.

[Show that $\int \lambda f d\mu = \lambda \int f d\mu$.]

First suppose that $\lambda \geq 0$. Then, $(\lambda f)^+ = \lambda f^+$ and $(\lambda f)^- =$ λf^-. Thus, $\int \lambda f d\mu = \int (\lambda f)^+ d\mu - \int (\lambda f)^- d\mu = \int \lambda f^+ d\mu$ $- \int \lambda f^- d\mu = \lambda \int f^+ d\mu - \lambda \int f^- d\mu = \lambda(\int f^+ d\mu - \int f^- d\mu)$ $= \lambda \int (f^+ - f^-)d\mu = \lambda \int f d\mu$. Next, if $\lambda < 0$, then $(\lambda f)^+ =$ $-\lambda f^-$ and $(\lambda f)^- = -\lambda f^+$. Thus, $\int \lambda f d\mu = \int (\lambda f)^+ d\mu -$ $\int (\lambda f)^- d\mu = \int -\lambda f^- d\mu - \int -\lambda f^+ d\mu = -\lambda \int f^- d\mu -$ $(-\lambda)\int f^+ d\mu = \lambda(\int f^+ d\mu - \int f^- d\mu) = \lambda \int (f^+ - f^-)d\mu$ $= \lambda \int f d\mu$.

Proof of Complex case is similar; use Re and Im

Proposition 2.22: If $f \in \mathcal{L}^1(\mu)$, then $|\int f d\mu| \leq \int |f| d\mu$.

Proof: If f is real-valued, then $|\int f d\mu| = |\int f^+ d\mu - \int f^- d\mu| \leq$ $\int f^+ d\mu + \int f^- d\mu = \int (f^+ + f^-)d\mu = \int |f| d\mu$. Next, suppose that f is complex-valued. If $\int f d\mu = 0$, then $|\int f d\mu|$ $\leq \int |f| d\mu$ is trivially true. So, suppose that $\int f d\mu \neq 0$. Let $\alpha = e^{i\theta}$ so that $\alpha \int f d\mu = |\int f d\mu|$. Then, $|\int f d\mu| = \alpha \int f d\mu$ $= \int \alpha f d\mu = \int Re(\alpha f) d\mu + i \int Im(\alpha f) d\mu = \int Re(\alpha f) d\mu <$ $\int |\alpha f| d\mu = \int |f| d\mu$ since $|\alpha| = |e^{i\theta}| = 1$.

Proposition 2.23

(a) If $f \in \mathcal{L}^1(\mu)$, then $\{x : f(x) \neq 0\}$ is σ-finite.

(b) Let $f, g \in \mathcal{L}^1(\mu)$. Then, $\int_E f d\mu = \int_E g d\mu$ for all $E \in \mathcal{M} \Leftrightarrow$ $\int |f - g| d\mu = 0 \Leftrightarrow f = g$ *a.e.* μ.

Proof of (a): Let $f \in \mathcal{L}^1(\mu)$, then $\int |f| d\mu < +\infty$. Then, $\{x : |f(x)| \neq 0\}$ $= \{x : f(x) \neq 0\} = \{x : f(x) > 0\}$ is σ-finite by Proposition 2.20.