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HW8: Let $f: \mathbb{R} \to \mathbb{R}$ be increasing, i.e., $x \leq y \Rightarrow f(x) \leq f(y)$.

Prove that f is $\mathcal{B}(\mathbb{R})$-measurable.

HW9: Let $f: \mathbb{R}^2 \to \mathbb{R}$ and suppose that for each $x \in \mathbb{R}$ the function $f_x : \mathbb{R} \to \mathbb{R}$, $f_x(y) = f(x,y)$ is $\mathcal{B}(\mathbb{R})$-measurable and that for each $y \in \mathbb{R}$ the function $f^y : \mathbb{R} \to \mathbb{R}$, $f^y(x) = f(x,y)$ is continuous. For each $n \in \mathbb{N}$ set $a_i = i/n$ and define

$$f_n(x,y) = \frac{f(a_{i+1}, y)(x - a_i) - f(a_i, y)(x - a_{i+1})}{a_{i+1} - a_i}$$

for $a_i \leq x \leq a_{i+1}$

Prove: (1) f_n is $\mathcal{B}(\mathbb{R}^2)$-measurable

(2) $\lim_n f_n(x,y) = f(x,y)$

(3) f is $\mathcal{B}(\mathbb{R}^2)$-measurable

(4) Give an example of a function $f: \mathbb{R}^2 \to \mathbb{R}$ such that f_x, f^y are continuous for each x, y but f is <u>not</u> continuous

Shorthand Notation: Let $(X, \mathcal{M})$ be a measurable space and $f : X \to \overline{\mathbb{R}}$. We will say f is measurable to mean f is $(\mathcal{M}, \mathcal{B}_{\overline{\mathbb{R}}})$-measurable.

Definition: Suppose that $f : X \to \overline{\mathbb{R}}$. Define $f^+ = max\{f, 0\}$ and $f^- = max\{-f, 0\}$. That is:

$$f^+(x) = \begin{cases} f(x) & \text{when } f(x) \geq 0 \\ 0 & \text{when } f(x) < 0 \end{cases}$$

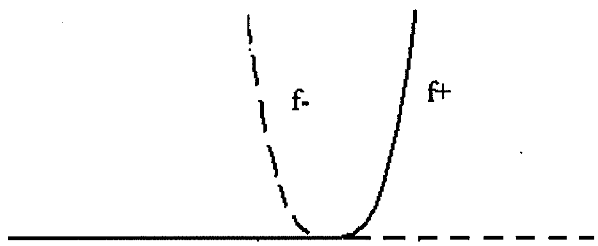
$$f^-(x) = \begin{cases} -f(x) & \text{when } f(x) \leq 0 \\ 0 & \text{when } f(x) > 0 \end{cases}$$

Note that:

(1) $f(x) = f^+(x) - f^-(x)$

(2) If f is measurable, then f^+ and f^- are both measurable by Corollary 2.8.

(3) $f^+(x) \cdot f^-(x) = 0$ because if $f^+ \neq 0$, then $f^- = 0$, and if $f^+ = 0$, then $f^- \neq 0$



Definition: Let $(X, \mathcal{M})$ be a measurable space. Then, $f : X \to \mathbb{R}$ is called a **simple function** if f is measurable and the range of f is a finite set.

Note that:

(1) If f is simple, then the range of f is finite, say $\{a_1, a_2, \ldots, a_n\}$. Then, $f^{-1}(\{a_i\}) = E_i \in \mathcal{M}$ and $f(x) = \sum_{i=1}^{n} a_i \chi_{E_i}(x)$ is called standard form. E_i's are disjoint.

(2) If $F_j \in \mathcal{M}$, let $f = \sum_{j=1}^{m} b_j \chi_{F_j}$, then f is simple. But, f may not be in <u>standard form</u>.

Example: $2 \cdot \chi_{[0,2]} + 3 \cdot \chi_{[1,3]}$ is not in standard form because $[0,2] \cap [1,3] \neq \emptyset$.
$2 \cdot \chi_{[0,1)} + 5 \cdot \chi_{[1,2]} + 3 \cdot \chi_{(2,3]}$ is in standard form because $[0,1) \cap [1,2] \cap (2,3] = \emptyset$.

~~(3) Every function is a limit of simple functions.~~

Theorem 2.10 (a): Let $(X, \mathcal{M})$ be a measurable space. If $f : X \to [0, +\infty]$ is $\mathcal{M}$-measurable, then there exist simple functions $\{\phi_n\}$ such that $0 \leq \phi_0 \leq \phi_1 \leq \cdots \leq f$ with $\lim_{n\to\infty} \phi_n(x) = f(x)$ for all x. If f is bounded on a set E, then $\phi_n \to f$ uniformly on E.

Proof: For $n_k = 0, 1, 2, \ldots$, and $0 \leq k \leq 2^{2n} - 1$, let $E_n^k = \{x : k/2^n < f(x) \leq (k+1)/2^n\}$, and $F_n = \{x : f(x) > 2^n\}$.
Define $\phi_n = \sum_{k=0}^{2^{2n}-1} k/2^n \cdot \chi_{E_n^k} + 2^n \cdot \chi_{F_n}$. It is clear that each ϕ_n is simple, and $\phi_n \geq 0$. Given any x with $f(x) \neq +\infty$, there exists N such that $f(x) \leq 2^N < 2^n$ for all $n \geq N \Rightarrow x \notin F_n$ for all $n > N$, and so there exists k such that $k/2^n < f(x) \leq (k+1)/2^n \Rightarrow \phi_n(x) = k/2^n \Rightarrow f(x) \geq \phi_n(x)$ and also $|f(x) - \phi_n(x)| \leq 1/2^n \Rightarrow \lim_n \phi_n(x) = f(x)$. On the other hand, if $f(x) = +\infty \Rightarrow f(x) > 2^n$ for all $n \Rightarrow x \in F_n$ for all $n \Rightarrow \phi_n(x) = 2^n \Rightarrow \lim_n \phi_n(x) = +\infty = f(x)$. Thus, $\phi_n \to f$ pointwise. Suppose that f is bounded on some set E. Then, there exists M such that $f(x) \leq M$ for all $x \in E$. Now, pick N such that $M < 2^N \Rightarrow f(x) < 2^n$ for all $x \in E$ and for all $n > N \Rightarrow |f(x) - \phi_n(x)| \leq 1/2^n$ for all $x \in E$. Thus, $\phi_n \to f$ uniformly on E.
To see ϕ is monotone increasing, consider the division into 2^n going to 2^{n+1}. First, $\phi_0 = 0 \cdot \chi_{E_0^0} + 2^0 \cdot \chi_{F_0}$ where $E_0^0 = \{x : 0/2^0 < f(x) \leq 1/2^0 = 1\}$ and $F_0 = \{x : f(x) > 1\}$.
Next, $\phi_1 = 0 \cdot \chi_{E_1^0} + 1/2 \cdot \chi_{E_1^1} + 2/2 \cdot \chi_{E_1^2} + 3/2 \cdot \chi_{E_1^3} + 2^1 \cdot F_1$ where $E_1^k = \{x : k/2^1 < f(x) \leq (k+1)/2^1\}$ such that $0 \leq k \leq 3$ and $F_1 = \{x : f(x) > 2^1\}$. We see that $\phi_0 \leq \phi_1$ as some of the range values of ϕ_0 pushed up to be larger in ϕ_1.

2.2 Integration of Non-Negative Functions

Idea: Let $(X, \mathcal{M}, \mu)$ be a measure space and $f : X \to [0, +\infty]$ be $\mathcal{M}$-measurable. We want to define $\int f d\mu =$ "area under graph"

If $\phi = \sum_{k=1}^{n} a_k \chi_{E_k}$, in standard form, define $\int \phi d\mu = \sum_{k=1}^{n} a_k \mu(E_k)$.

By Theorem 2.10, there exist $0 \leq \phi_0 \leq \phi_1 \leq \cdots \leq f$, and we would like $\int f d\mu = \lim_n \int \phi_n d\mu$.

Fix a measure space $(X, \mathcal{M}, \mu)$, and let $L^+ = \{f : X \to [0, +\infty] : f$ is $\mathcal{M}$-measurable.$\}$

Definition: Let $\phi \in L^+$ be simple with standard representation $\phi = \sum_{k=1}^{n} a_k \chi_{E_k}$. Define $\int \phi d\mu = \sum_{k=1}^{n} a_k \mu(E_k)$. For $A \in \mathcal{M}$, we define $\int_A \phi d\mu = \sum_{k=1}^{n} a_k \mu(E_k \cap A) = \int \phi \chi_A d\mu$.

Proposition 2.13: Let $\phi, \psi \in L^+$ be simple. Then:

(a) If $c \geq 0$, then $\int (c\phi) d\mu = c \int \phi d\mu$.

(b) $\int (\phi + \psi) d\mu = \int \phi d\mu + \int \psi d\mu$

(c) If $0 \leq \phi \leq \psi$, then $\int \phi d\mu \leq \int \psi d\mu$.

(d) Fix ϕ and define $\nu(A) = \int_A \phi d\mu$, then $\nu : \mathcal{M} \to [0, +\infty]$ is also a measure.

Proof of (a): If $c = 0$, then both sides equal to 0. Let $c > 0$, then $\phi = \sum_{j=1}^{n} a_j \chi_{E_j}$, in standard form, and $c\phi = \sum_{j=1}^{n} (ca_j) \chi_{E_j}$ which is the standard form of $c\phi$. Thus, $\int (c\phi) d\mu = \sum_{j=1}^{n} (ca_j) \mu(E_j) = c \sum_{j=1}^{n} a_j \mu(E_j) = c \int \phi d\mu$.

Proof of (b): Let $\phi = \sum_{j=1}^{n} a_j \chi_{E_j}$ and $\psi = \sum_{k=1}^{m} b_k \chi_{F_k}$ be both in standard form. Then, $E_1 \dot\cup E_2 \dot\cup \cdots \dot\cup E_n = X = F_1 \dot\cup F_2 \dot\cup \cdots \dot\cup F_m$. This implies that $E_j \cap F_k$ are disjoint. So, $\bigcup_{j=1}^{n} \bigcup_{k=1}^{m} (E_j \cap F_k) = X$.

Also, note that $\bigcup_{k=1}^{m} E_j \cap F_k = E_j$ and $\bigcup_{j=1}^{n} E_j \cap F_k = F_k$.

Thus, $\int \phi d\mu + \int \psi d\mu = \sum_{j=1}^{n} a_j \mu(E_j) + \sum_{k=1}^{m} b_k \mu(F_k) =$

$\sum_{j=1}^{n}\sum_{k=1}^{m} a_j \mu(E_j \cap F_k) + \sum_{k=1}^{m}\sum_{j=1}^{n} b_k (F_k \cap E_j) =$

$\sum_{j=1}^{n}\sum_{k=1}^{m} (a_j + b_k)\mu(E_j \cap F_k)$. Now, let $\phi + \psi = \sum_l c_l \chi_{G_l}$, and let $x \in G_l$. Then, there exist j_0 and k_0 such that $x \in E_{j_0} \cap F_{k_0}$. This implies that $c_l = (\phi + \psi)(x) = a_{j_0} + b_{k_0}$. Thus, each c_l is a sum $a_j + b_k$ and $G_l = \bigcup_{j,k}\{E_j \cap F_k : c_l = a_j + b_k\}$. Thus, $\mu(G_l)$ $= \sum_{\substack{j,k \\ a_j+b_k=c_l}} \mu(E_j \cap F_k)$, and so $\int (\phi + \psi) d\mu = \sum_l c_l \mu(G_l) =$

$\sum_l (a_j + b_k) \sum_{\substack{j,k \\ a_j+b_k=c_l}} \mu(E_j \cap F_k) = \sum_{j=1}^{n}\sum_{k=1}^{m} (a_j + b_k)\mu(E_j \cap F_k) =$

$\int \phi d\mu + \int \psi d\mu$.

Proof of (c): Let $0 \le \phi \le \psi$. Then, $\psi = \phi + (\psi - \phi)$ where $\psi - \phi$ is still simple. Thus, by (b), $\int \psi d\mu = \int [\phi + (\psi - \phi)] d\mu =$ $\int \phi d\mu + \int (\psi - \phi) d\mu \ge \int \phi d\mu$.

Proof of (d): Fix ϕ and let $\phi = \sum_{j=1}^{n} a_j \chi_{E_j}$, in standard form, and define $\nu : \mathcal{M} \to [0, +\infty]$ by $\nu(A) = \int_A \phi d\mu$. It is clear that $\nu(\emptyset) =$ $\int_\emptyset \phi d\mu = 0$ and $\nu(A) = \int_A \phi d\mu \ge 0$. Let $A = \bigcup_k A_k$ where each A_k are disjoint. Then, $\nu(\bigcup_k A_k) = \nu(A) = \int_A \phi d\mu =$

$\sum_{j=1}^{n} a_j \mu(E_j \cap A) = \sum_{j=1}^{n} a_j [\sum_{k=1}^{\infty} \mu(E_j \cap A_k)] =$

$\sum_{k=1}^{\infty} [\sum_{j=1}^{n} a_j \mu(E_j \cap A_k)] = \sum_{k=1}^{\infty} [\int_{A_k} \phi d\mu] = \sum_{k=1}^{\infty} \nu(A_k)$.

Thus, ν is a measure.

Corollary: Let ϕ be simple and $\phi = \sum_{l=1}^{m} b_l \chi_{F_l}$, may not be in standard form, but each $b_l \geq 0$. Then, $\int \phi d\mu = \sum_{l=1}^{m} b_l \mu(F_l)$.

Proof: By Proposition 2.14-(b) and induction, $\int \phi d\mu = \sum_{l=1}^{m} [\int b_l \chi_{F_l} d\mu]$

$= \sum_{l=1}^{m} \int (b_l \chi_{F_l} + 0 \cdot \chi_{F_l^c}) d\mu$ (notice that this is in standard form)

$= \sum_{l=1}^{m} b_l \mu(F_l) + 0 \cdot \mu(F_l^c) = \sum_{l=1}^{m} b_l \mu(F_l)$.

Definition: Let $(X, \mathcal{M}, \mu)$ be a measure space. For, $f \in L^+$, we define $\int f d\mu = sup\{\int \phi d\mu : 0 \leq \phi \leq f$ and ϕ is simple.$\}$

~~Note~~ Comp. (1): Let ψ be simple. Then. $\int \psi d\mu = sup\{\int \phi d\mu : 0 \leq \phi \leq \psi$ and ϕ is simple. We have no trouble here, sup gives the same value for a simple function ψ.

Note (2): If $f \in L^+$ and $c > 0$, then $\int cf d\mu = sup\{\int \phi d\mu : 0 \leq \phi \leq cf$ and ϕ is simple.$\}$ Let $\psi = c^{-1}\phi$, that is $c\psi = \phi$, where ψ is simple. Then, $\phi \leq cf$ and ϕ is simple $\Leftrightarrow c\psi \leq cf$ and ψ is simple $\Leftrightarrow \psi \leq f$ and ψ is simple. So, $\int cf d\mu = sup\{\int \phi d\mu : 0 \leq \phi \leq cf$ and ϕ is simple.$\} =$

$sup\{\int \phi d\mu : 0 \leq \psi \leq f, \psi$ is simple and $\psi = c^{-1}\phi.\} =$

$sup\{\int c\psi d\mu : 0 \leq \psi \leq f$, and ψ is simple.$\} =$

$c sup\{\int \psi d\mu : 0 \leq \psi \leq f$, and ψ is simple.$\} =$

$c \int f d\mu$.

Prop'n ~~Note (3)~~: If $f, g \in L^+$ and $f \leq g$, then $\int f d\mu \leq \int g d\mu$ because there are more ϕ's in the definition of sup for $\int g d\mu$.

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Theorem (The Monotone Convergence Theorem): Suppose that $\{f_n\} \subseteq L^+$ and $f_j \leq f_{j+1}$ for all j. Let $f(x) = \sup_j f_j(x) = \lim_j f(x)$.

Then, $\int f d\mu = \lim_j \int f_j d\mu = \sup_j \int f_j d\mu$.

Proof: [Show that $\sup_j \int f_j d\mu \leq \int f d\mu$.]

Since each $f_j \leq f$, $\int f_j d\mu \leq \int f d\mu \Rightarrow \sup_j \int f_j d\mu \leq \int f d\mu$.