$\mathcal{B}_X$ written for $\mathcal{B}(X)$

Corollary 2.2: Let X and Y be topological spaces (or metric spaces), $\mathcal{B}(X)$ and $\mathcal{B}(Y)$ be Borel sets. If $f : X \to Y$ is continuous, then f is $(\mathcal{B}(X), \mathcal{B}(Y))$-measurable.

Proof: Let X and Y be topological spaces (or metric spaces), $\mathcal{B}_X$ and $\mathcal{B}_Y$ be Borel sets and $f : X \to Y$ be continuous. Let $\mathcal{E}$ be the collection of open subsets of Y. Then, $\mathcal{E}$ generates $\mathcal{B}_Y$. Let $U \in \mathcal{E}$, then $f^{-1}(U)$ is open in X because f is continuous. This implies that $f^{-1}(U) \in \mathcal{B}_X$. Thus, by Proposition 2.1, f is $(\mathcal{B}_X, \mathcal{B}_Y)$- measurable.

Shorthand: Let $(X, \mathcal{M})$ be a measurable space and $f : X \to \mathbb{R}$, then we say that f is $\mathcal{M}$-measurable or measurable to mean that f is $(\mathcal{M}, \mathcal{B}_\mathbb{R})$-measurable. That is $f^{-1}(B) \in \mathcal{M}$ for B, the Borel sets of $\mathbb{R}$.

Proposition 2.3: Let $(X, \mathcal{M})$ be a measurable space and $f : X \to \mathbb{R}$. Then the following statements are equivalent:

(a) f is $\mathcal{M}$-measurable.
(b) $f^{-1}((a, +\infty)) \in \mathcal{M}$ for all $a \in \mathbb{R}$.
(c) $f^{-1}([a, +\infty)) \in \mathcal{M}$ for all $a \in \mathbb{R}$.
(d) $f^{-1}((-\infty, a)) \in \mathcal{M}$ for all $a \in \mathbb{R}$.
(e) $f^{-1}((-\infty, a]) \in \mathcal{M}$ for all $a \in \mathbb{R}$.

Proof: ((a) $\Rightarrow$ (b)) Suppose that f is $(\mathcal{M}, \mathcal{B}_\mathbb{R})$-measurable. Then, $(a, +\infty) \in \mathcal{B}_\mathbb{R}$ implies that $f^{-1}((a, +\infty)) \in \mathcal{M}$.
((b) $\Rightarrow$ (a)) We have shown earlier that the sets of the form $\mathcal{E} = \{(a, +\infty) : a \in \mathbb{R}\}$ generate $\mathcal{B}_\mathbb{R}$. Thus, by Proposition 2.1, f is $(\mathcal{M}, \mathcal{B}_\mathbb{R})$-measurable.
Note: The rest of the proofs are similar. Use that fact that the given collections generate the Borel sets.

Composition

Note 1: Suppose that $f, g : \mathbb{R} \to \mathbb{R}$, and they are both $\mathcal{B}_\mathbb{R}$-measurable. This means that if $B \in \mathcal{B}_\mathbb{R}$, then $f^{-1}(B), g^{-1}(B) \in \mathcal{B}_\mathbb{R}$. Then, if $f \circ g : \mathbb{R} \to \mathbb{R}$ and $B \in \mathcal{B}_\mathbb{R}$, then $(f \circ g)^{-1}(B) = g^{-1}(f^{-1}(B)) \in \mathcal{B}_\mathbb{R}$. Thus, $f \circ g$ is $\mathcal{B}_\mathbb{R}$-measurable.

Note 2: Let $\mathcal{L}$ be the collection of Lebesgue sets. Suppose that $f, g : \mathbb{R} \to \mathbb{R}$ and they are both $\mathcal{L}$-measurable. That is if $B \in \mathcal{B}_\mathbb{R}$, then $f^{-1}(B), g^{-1}(B) \in \mathcal{L}$. Now consider $f \circ g : \mathbb{R} \to \mathbb{R}$ and let $B \in \mathcal{B}_\mathbb{R}$. Now, $(f \circ g)^{-1}(B) = g^{-1}(f^{-1}(B))$ and $f^{-1}(B) \in \mathcal{L}$, but $f^{-1}(B)$ may not be a Borel set. Thus, in general, $(f \circ g)^{-1}(B) = g^{-1}(f^{-1}(B)) \notin \mathcal{L}$. That is, $f \circ g$ need not be $\mathcal{L}$-measurable. In fact such examples exist.

Definition: Let $E \subseteq X$. Then the function $\chi_E : X \to \mathbb{R}$ defined by
$$\chi_E(x) = \begin{cases} 1 & \text{if } x \in E \\ 0 & \text{if } x \notin E \end{cases}$$
is called the **characteristic function** of the set E.

Proposition: Let $(X, \mathcal{M})$ be a measurable space, and $E \subseteq X$. Then, $\chi_E : X \to \mathbb{R}$ is $\mathcal{M}$-measurable if and only if $E \in \mathcal{M}$.

Proof: χ_E is measurable $\Leftrightarrow \chi_E^{-1}(B) \in \mathcal{M}$ for all $B \in \mathcal{B}_{\mathbb{R}}$. But,
$$\chi_E^{-1}(B) = \begin{cases} \emptyset, & 0, 1 \notin B \\ E & 1 \in B,\ 0 \notin B \\ E^c, & 0 \in B,\ 1 \notin B \\ X, & \{0, 1\} \subseteq B \end{cases}$$
Thus, χ_E is measurable $\Leftrightarrow E, E^c \in \mathcal{M}$
$\Leftrightarrow E \in \mathcal{M}$.

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Example: Recall $h(x) = x + f(x)$ where f is ternary, and $h : [0.1] \to [0.2]$ is a homeomorphism. If $C \subseteq [0, 1]$ is a Cantor set, then $h(C)$ has measure 1.

So, there exist $E \subseteq h(C)$ such that E is non-measurable. Pick E such that $0 \notin E$ and $2 \notin E$. Let $B = h^{-1}(E) \subseteq C$. Then, B is measurable, $0 \notin B$ and $1 \notin B$. So, $h^{-1} : [0, 2] \to [0, 1]$ is continuous and can be extended to $b : \mathbb{R} \to \mathbb{R}$ defined by
$$b(x) = \begin{cases} 0 & \text{if } x < 0 \\ h^{-1}(x) & \text{if } 0 \leq x \leq 2 \\ 1 & \text{if } 2 < x \end{cases}$$
Thus, b is continuous on $\mathbb{R}$ implies that b is $(\mathcal{B}_{\mathbb{R}}, \mathcal{B}_{\mathbb{R}})$-measurable. Note that χ_B is $(\mathcal{L}, \mathcal{B}_{\mathbb{R}})$-measurable. Now, consider $(\chi_B \circ b)^{-1}(\{1\}) = \{x : (\chi_B \circ b)(x) = 1\} = \{x : b(x) \in B\} = \{x : h^{-1}(x) \in B\} = (h^{-1})^{-1}(B) = h(B) = E$ which is non-measurable. Thus, $\chi_B \circ b$ is not $(\mathcal{L}, \mathcal{B}_{\mathbb{R}})$-measurable.

Idea of Finite Products: Given measurable spaces $(Y_1, \mathcal{N}_1)$ and $(Y_2, \mathcal{N}_2)$. We want a σ-algebra on the product space $Y_1 \times Y_2$. To define such a σ-algebra, look at the σ-algebra generated by all sets of the form $\{B_1 \times B_2 : B_1 \in \mathcal{N}_1 \text{ and } B_2 \in \mathcal{N}_2\}$. This σ-algebra is denoted by $\mathcal{N}_1 \otimes \mathcal{N}_2$.

Proposition: Let $Y_1 = Y_2 = \mathbb{R}$ and $\mathcal{N}_1 = \mathcal{N}_2 = \mathcal{B}_{\mathbb{R}}$. Then, $Y_1 \times Y_2 = \mathbb{R}^2$ and $\mathcal{B}_{\mathbb{R}} \otimes \mathcal{B}_{\mathbb{R}} = \mathcal{B}_{\mathbb{R}^2}$.

Proof: [Show that $\mathcal{B}_{\mathbb{R}^2} \subseteq \mathcal{B}_{\mathbb{R}} \otimes \mathcal{B}_{\mathbb{R}}$.]
Let $O \in \mathbb{R}^2$ be open. Look at rectangles $(p_1, q_1) \times (p_2, q_2) \subseteq O$ such that $p_i, q_i \in \mathbb{Q}$. Since O is open, $O = \cup\{(p_1, q_1) \times (p_2, q_2) : p_i, q_i \in \mathbb{Q}$ and $(p_1, q_1) \times (p_2, q_2) \subseteq O\}$. Then, this is a countable union. But, $(p_i, q_i) \in \mathcal{B}_{\mathbb{R}} \Rightarrow (p_1, q_1) \times (p_2, q_2) \in \mathcal{B}_{\mathbb{R}} \otimes \mathcal{B}_{\mathbb{R}} \Rightarrow O \in \mathcal{B}_{\mathbb{R}} \otimes \mathcal{B}_{\mathbb{R}}$. Since open sets generates $\mathcal{B}_{\mathbb{R}^2}$, $\mathcal{B}_{\mathbb{R}^2} \subseteq \mathcal{B}_{\mathbb{R}} \otimes \mathcal{B}_{\mathbb{R}}$.
[Show that $\mathcal{B}_{\mathbb{R}} \otimes \mathcal{B}_{\mathbb{R}} \subseteq \mathcal{B}_{\mathbb{R}^2}$.]
It is enough to show that if $B_1, B_2 \in \mathcal{B}_{\mathbb{R}}$, then $B_1 \times B_2 \in \mathcal{B}_{\mathbb{R}^2}$.
Let $\mathcal{A} = \{E \subseteq \mathbb{R} : E \times \mathbb{R} \in \mathcal{B}_{\mathbb{R}^2}\}$. Note that $\emptyset \in \mathcal{A}$, $\mathbb{R} \in \mathcal{A}$, and $E \in \mathcal{A} \Rightarrow E \times \mathbb{R} \in \mathcal{B}_{\mathbb{R}^2} \Rightarrow (E \times \mathbb{R})^c \in \mathcal{B}_{\mathbb{R}^2}$. But, $(E \times \mathbb{R})^c = E^c \times \mathbb{R} \Rightarrow E^c \in \mathcal{A}$.
Finally, $E_n \in \mathcal{A} \Rightarrow E_n \times \mathbb{R} \in \mathcal{B}_{\mathbb{R}^2} \Rightarrow \bigcup\limits_n (E_n \times \mathbb{R}) \in \mathcal{B}_{\mathbb{R}^2}$. But, $\bigcup\limits_n (E_n \times \mathbb{R}) = (\bigcup\limits_n E_n) \times \mathbb{R} \Rightarrow \bigcup\limits_n E_n \in \mathcal{A}$.
Thus, $\mathcal{A}$ *is a* σ-algebra.
If we do the following: $(a, b) \times \mathbb{R} \in \mathcal{B}_{\mathbb{R}^2} \Rightarrow (a, b) \in \mathcal{A}$. Thus, $\mathcal{B}_{\mathbb{R}} \subseteq \mathcal{A}$. Hence, if $B \in \mathcal{B}_{\mathbb{R}}$, then $B \times \mathbb{R} \in \mathcal{B}_{\mathbb{R}^2}$. Similarly, if $B \in \mathcal{B}_{\mathbb{R}}$, then $\mathbb{R} \times B \in \mathcal{B}_{\mathbb{R}^2}$. Hence, $B_1 \times B_2 = (B_1 \times \mathbb{R}) \cap (\mathbb{R} \times B_2) \in \mathcal{B}_{\mathbb{R}^2}$. This shows that $\mathcal{B}_{\mathbb{R}} \otimes \mathcal{B}_{\mathbb{R}} \subseteq \mathcal{B}_{\mathbb{R}^2}$, and so $\mathcal{B}_{\mathbb{R}} \otimes \mathcal{B}_{\mathbb{R}} = \mathcal{B}_{\mathbb{R}^2}$.

Proposition 2.4: (Case $n = 2$) Given $(X, \mathcal{M})$, $(Y_1, \mathcal{N}_1)$, and $(Y_2, \mathcal{N}_2)$, and also $f_1 : X \to Y_1$ and $f_2 : X \to Y_2$, define $f : X \to Y_1 \times Y_2$ by $f(x) = (f_1(x), f_2(x))$. Then, f is $(\mathcal{M}, \mathcal{N}_1 \otimes \mathcal{N}_2)$-measurable if and only if f_i is $(\mathcal{M}, \mathcal{N}_i)$-measurable for each $i = 1, 2$.

Proof: $(\Rightarrow)$ Given $B_1 \in \mathcal{N}_1 \Rightarrow B_1 \times Y_2 \in \mathcal{N}_1 \times \mathcal{N}_2 \Rightarrow f^{-1}(B_1 \times Y_2) \in \mathcal{M}$. But, $f^{-1}(B_1 \times Y_2) = \{x : f(x) \in B_1 \times Y_2\} = \{x : (f_1(x), f_2(x)) \in B_1 \times Y_2\} = \{x : f_1(x) \in B_1\} = f_1^{-1}(B_1)$. Thus, $f_1^{-1}(B_1) \in \mathcal{M}$, and so f_1 is $(\mathcal{M}, \mathcal{N}_1)$-measurable. Similarly, f_2 is $(\mathcal{M}, \mathcal{N}_2)$-measurable.
$(\Leftarrow)$ Let $\mathcal{A} = \{B \in Y_1 \times Y_2 : f^{-1}(B) \in \mathcal{M}\}$. It is easy to show that $\mathcal{A}$ is a σ-algebra. Let $B_i \in \mathcal{N}_i$. Then, $B_1 \times B_2 = (B_1 \times Y_2) \cap (Y_1 \times B_2) \Rightarrow f^{-1}(B_1 \times B_2) = f^{-1}(B_1 \times Y_2) \cap f^{-1}(Y_1 \times B_2)$. Now, $f^{-1}(B_1 \times Y_2) = \{x : f(x) \in B_1 \times Y_2\} = \{x : f_1(x) \in B_1\} = f_1^{-1}(B_1) \in \mathcal{M} \Rightarrow B_1 \times Y_2 \in \mathcal{A}$, and

similarly $Y_1 \times B_2 \in \mathcal{A} \Rightarrow B_1 \times B_2 = (B_1 \times Y_2) \cap (Y_1 \times B_2)$ $\in \mathcal{A}$. Thus, $\mathcal{N}_1 \otimes \mathcal{N}_2 \subseteq \mathcal{A}$, and so $f^{-1}(E) \in \mathcal{M}$ for all $E \in \mathcal{N}_1 \otimes \mathcal{N}_2$. Therefore, f is $(\mathcal{M}, \mathcal{N}_1 \otimes \mathcal{N}_2)$-measurable.

Proposition 2.6: Let $(X, \mathcal{M})$ be measurable. If $f, g : X \to \mathbb{R}$ are $(\mathcal{M}, \mathcal{B}_{\mathbb{R}})$-measurable, then $f+g$ and fg are $(\mathcal{M}, \mathcal{B}_{\mathbb{R}})$-measurable.

Proof: Look at $F : X \to \mathbb{R}^2$ defined by $F(x) = (f(x), g(x))$. By Proposition 2.4, F is $(\mathcal{M}, \mathcal{B}_{\mathbb{R}} \otimes \mathcal{B}_{\mathbb{R}})$-measurable.
Let $s : \mathbb{R}^2 \to \mathbb{R}$ be defined by $s((x,y)) = x + y$. Then, s is continuous. This implies that s is $(\mathcal{B}_{\mathbb{R}^2}, \mathcal{B}_{\mathbb{R}})$-measurable by Corollary 2.2. But, $\mathcal{B}_{\mathbb{R}} \otimes \mathcal{B}_{\mathbb{R}} = \mathcal{B}_{\mathbb{R}^2}$. So, $s \circ F : X \to \mathbb{R}$ is $(\mathcal{M}, \mathcal{B}_{\mathbb{R}})$-measurable, and $(s \circ F)(x) = s((f(x), g(x)) = f(x) + g(x)$. Thus, $f + g$ is $(\mathcal{M}, \mathcal{B}_{\mathbb{R}})$-measurable.
Next, $p : \mathbb{R}^2 \to \mathbb{R}$ be defined by $p((a,b)) = ab$. Then, p is continuous. By Corollary 2.2, p is $(\mathcal{B}_{\mathbb{R}^2}, \mathcal{B}_{\mathbb{R}})$-measurable. So, $p \circ F : X \to \mathbb{R}$ is $(\mathcal{M}, \mathcal{B}_{\mathbb{R}})$-measurable , and $(p \circ F)(x) = p(f(x), g(x)) = f(x)g(x)$. Thus, fg is $(\mathcal{M}, \mathcal{B}_{\mathbb{R}})$-measurable.

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<u>General Product</u>

Let $(Y_\alpha, \mathcal{N}_\alpha)$ where $\alpha \in A$ be measurable spaces. Then, $Y = \prod\limits_{\alpha \in A} Y_\alpha$ $= \{y : y = (y_\alpha)_{\alpha \in A} \text{ and } y_\alpha \in Y_\alpha\}$. Also, the **product σ-algebra**, denoted by $\mathcal{N} = \bigotimes\limits_{\alpha \in A} \mathcal{N}_\alpha$, is the σ-algebra generated by all sets of the following form:

Pick countably many α's, say $\{\alpha_n\}$.
For each n, pick $E_n \in \mathcal{N}_{\alpha_n}$.
Let $E = \{y = (y_\alpha) : y_{\alpha_n} \in E_n \text{ for all } n\}$ which is called **countable windows.**

Another set that generates $\mathcal{N}$:

Pick one α, say α_1 and $E_1 \in \mathcal{N}_{\alpha_1}$.
Let $E = \{y = (y_\alpha) : y_{\alpha_1} \in E_1\}$.

They are the same set because:

$$\{y = (y_\alpha) : y_{\alpha_n} \in E_n \text{ for all } n\} = \bigcap_{n=1}^{\infty} \{\{y = (y_\alpha) : y_{\alpha_n} \in E_n\}.$$

Proposition 2.4: Let $(X, \mathcal{M})$ and $(Y_\alpha, \mathcal{N}_\alpha)$ be measurable spaces, and $f_\alpha : X \to Y_\alpha$. Define $f : X \to \prod Y_\alpha$ by $f(x) = (f_\alpha(x))$. Then, f is $(\mathcal{M}, \bigotimes \mathcal{N}_\alpha)$-measurable if and only if f_α is $(\mathcal{M}, \mathcal{N}_\alpha)$-measurable for all α.

<u>Extended Reals</u>

$\overline{\mathbb{R}}_e = \mathbb{R} \cup \{+\infty\} \cup \{-\infty\}$

$\mathcal{B}(\mathbb{R}_e)$ $\mathcal{B}_{\overline{\mathbb{R}}_e} = \{B \subseteq \overline{\mathbb{R}}_e : B \cap \mathbb{R} \in \mathcal{B}_{\mathbb{R}}\}$

Note that $\mathcal{B}_{\overline{\mathbb{R}}}$ is a σ-algebra, and $B \in \mathcal{B}_{\overline{\mathbb{R}}}$ if and only if $B = B_1$, $B_1 \cup \{+\infty\}$, $B_1 \cup \{-\infty\}$, or $B_1 \cup \{+\infty\} \cup \{-\infty\}$ where $B_1 \in \mathcal{B}_{\mathbb{R}}$.

Proposition: Let $(X, \mathcal{M})$ be a measurable space and $f : X \to \overline{\mathbb{R}}_e$ Then, following statements are equivalent:

(1) f is ~~$(\mathcal{M}, \mathcal{B}_{\overline{\mathbb{R}}})$-measura~~ble. $(\mathcal{M}, \mathcal{B}(\mathbb{R}_e))$-measurable

(2) $f^{-1}((a, +\infty]) \in \mathcal{M}$ for all $a \in \mathbb{R}$

(3) $f^{-1}([a, +\infty]) \in \mathcal{M}$ for all $a \in \mathbb{R}$

(4) $f^{-1}([-\infty, a)) \in \mathcal{M}$ for all $a \in \mathbb{R}$

(5) $f^{-1}([-\infty, a]) \in \mathcal{M}$ for all $a \in \mathbb{R}$

Proof: ((1) $\Rightarrow$ (2)) Recall that $(a, +\infty] = (a, +\infty) \cup \{+\infty\}$, and $(a, +\infty) \in \mathcal{B}_{\mathbb{R}}$ implies that $(a, +\infty) \cup \{+\infty\} \in \mathcal{B}_{\overline{\mathbb{R}}}$. Since f is $(\mathcal{M}, \mathcal{B}_{\overline{\mathbb{R}}})$-measurable, $f^{-1}((a, +\infty]) \in \mathcal{M}$ for all $a \in \mathbb{R}$. Similarly, (1) $\Rightarrow$ (3), (1) $\Rightarrow$ (4), and (1) $\Rightarrow$ (5).

((2) $\Rightarrow$ (1) - sketch) The set of the form $(a, +\infty]$ generates $\mathcal{B}_{\overline{\mathbb{R}}}$, and $(a, +\infty]^c = \{-\infty\}$. Now, $\{B : f^{-1}(B) \in \mathcal{M}\}$ is a σ-algebra, and a generating set for $\mathcal{B}_{\overline{\mathbb{R}}}$. So, it must contain $\mathcal{B}_{\overline{\mathbb{R}}}$. Thus, $B \in \mathcal{B}_{\overline{\mathbb{R}}}$, then $f^{-1}(B) \in \mathcal{M}$, and so f is $(\mathcal{M}, \mathcal{B}_{\overline{\mathbb{R}}})$-measurable.

Similarly, (3) $\Rightarrow$ (1), (4) $\Rightarrow$ (1), and (5) $\Rightarrow$ (1) because $[a, +\infty]$, $[-\infty, a)$ and $[-\infty, a]$ also generate $\mathcal{B}_{\overline{\mathbb{R}}}$.

Proposition 2.7: Let $(X, \mathcal{M})$ be a measurable space, and $f_j : X \to \overline{\mathbb{R}}$ be all $(\mathcal{M}, \mathcal{B}_{\overline{\mathbb{R}}})$-measurable. Then, $g_1(x) = \sup_j f_j(x)$, $g_2(x) = \inf_j f_j(x)$, $g_3(x) = \lim_j \sup_j f_j(x)$ and $g_4(x) = \lim_j \inf_j f_j(x)$ are all $(\mathcal{M}, \mathcal{B}_{\overline{\mathbb{R}}})$-measurable. If $\lim_j f_j(x)$ exists for all x, then it is also $(\mathcal{M}, \mathcal{B}_{\overline{\mathbb{R}}})$-measurable.

Proof: [Show that g_1 is $(\mathcal{M}, \mathcal{B}_{\overline{\mathbb{R}}})$-measurable.]

$g_1^{-1}((a, +\infty]) = \bigcup_{j=1}^{\infty} f_j^{-1}((a, +\infty]) \in \mathcal{M}$ because each $f_j^{-1}((a, +\infty]) \in \mathcal{M}$. Thus, g_1 is $(\mathcal{M}, \mathcal{B}_{\overline{\mathbb{R}}})$-measurable.

[Show that g_2 is g_1 is $(\mathcal{M}, \mathcal{B}_{\overline{\mathbb{R}}})$-measurable.]

$g_2^{-1}([-\infty, a)) = \bigcup_{j=1}^{\infty} f_j^{-1}([-\infty, a)) \in \mathcal{M}$ because each $f_j^{-1}([-\infty, a)) \in \mathcal{M}$. Thus, g_2 is $(\mathcal{M}, \mathcal{B}_{\overline{\mathbb{R}}})$-measurable.

[Show that g_3 is $(\mathcal{M}, \mathcal{B}_{\overline{\mathbb{R}}})$-measurable.]
Let $h_k(x) = \sup_{j>k} f_j(x)$, then $\inf_k h_k(x) = \lim_j \sup f_j(x)$. By the first argument, h_k is measurable for all k. This implies that $\inf_k h_k(x) = \lim_j \sup f_j(x) = g_3(x)$ is measurable. Thus, g_3 is $(\mathcal{M}, \mathcal{B}_{\overline{\mathbb{R}}})$-measurable.
[Show that g_4 is $(\mathcal{M}, \mathcal{B}_{\overline{\mathbb{R}}})$-measurable.]
Similarly, let $l_k(x) = \inf_{j>k} f_j(x)$, then $\sup_k l_k(x) = \lim_j \inf f_j(x)$ $= g_4(x)$. Thus, g_4 is $(\mathcal{M}, \mathcal{B}_{\overline{\mathbb{R}}})$-measurable.
Finally, if $\lim_j f_j(x)$ exists for all x, the it is the same as $\lim_j \sup f_j(x) = \lim_j \inf f_j(x)$, and so $\lim_j f_j(x)$ is $(\mathcal{M}, \mathcal{B}_{\overline{\mathbb{R}}})$-measurable.

Corollary 2.8: Let $(X, \mathcal{M})$ be a measurable space, and $f, g : X \to \overline{\mathbb{R}}$ be $(\mathcal{M}, \mathcal{B}_{\overline{\mathbb{R}}})$-measurable. Then, $max\{f, g\}$ and $min\{f, g\}$ are both $(\mathcal{M}, \mathcal{B}_{\overline{\mathbb{R}}})$-measurable.

<u>Note</u>: Let $(X, \mathcal{M})$ be a measurable space and $E \in \mathcal{M}$. Then, $\mathcal{M}_E = \{B \cap E : B \in \mathcal{M}\}$ is a σ-algebra of subsets of E.

Corollary 2.9$^+$(This says more than the text): Let $(X, \mathcal{M})$ be a measurable space, and suppose that $f_j : X \to \overline{\mathbb{R}}$ be all $(\mathcal{M}, \mathcal{B}_{\overline{\mathbb{R}}})$-measurable. Then, $E = \{x : \lim_j f_j(x) \text{ exists}\}$ is a measurable set. Moreover, if we define $f : E \to \overline{\mathbb{R}}$ by $f(x) = \lim_j f_j(x)$, then f is $(\mathcal{M}_E, \mathcal{B}_{\overline{\mathbb{R}}})$-measurable.

Proof: Let $E = \{x : \lim_j f_j(x) \text{ exists}\} = \{x : \lim_j \sup f_j(x) = \lim_j \inf f_j(x)\} = \{x : g_3(x) = g_4(x)\} = \{x : g_3(x) - g_4(x) = 0\} = (g_3 - g_4)^{-1}(\{0\})$. Since $g_3 - g_4$ is $(\mathcal{M}, \mathcal{B}_{\overline{\mathbb{R}}})$-measurable, $(g_3 - g_4)^{-1}(\{0\}) = E \in \mathcal{M}$, that is E is a measurable set.
On E, $\lim_j f_j(x) = g_3(x) = g_4(x)$. So, $\lim_j f_j(x)$ is measurable on E. Thus, if define $f : E \to \overline{\mathbb{R}}$ by $f(x) = \lim_j f_j(x)$, then f is $(\mathcal{M}_E, \mathcal{B}_{\overline{\mathbb{R}}})$-measurable.