Summary: Given $F: \mathbb{R} \to \mathbb{R}$, increasing and right continuous, set

$\mu_F((a,b]) = F(b) - F(a)$

yields outer measure μ_F^*, restrict to ~~Caratheodory~~ $\mathcal{M}_F = \{A \mid A\ \mu_F^*\text{-measurable}\}$

obtain by Caratheodory a measure called Lebesgue-Stieltjes measure, μ_F

Know: ① $\mu_F((a,b]) = \mu_F^*((a,b]) = F(b) - F(a)$

② $B(\mathbb{R}) \subseteq \mathcal{M}_F$.

③ μ_F is σ-finite, since $\mathbb{R} = \bigcup (n, n+1]$ all finite

When we take $F(x) = x$, corresponding Lebesgue-Stieltjes measure, called Lebesgue measure, denoted m.

In 450, construct a set E that is not Lebesgue measurable, omit in this course

Theorem 1.16: If $F:\mathbb{R}\to\mathbb{R}$ is increasing and right continuous, then there exists ~~unique~~ a **Borel measure** μ_F on $\mathbb{R}$ ~~such~~ with $\mu_F((a,b]) = F(b)-F(a)$. If G is another such function, then $\mu_F=\mu_G$ if and only if $F-G=$ a constant. [Note that such a measure is bounded on bounded intervals.] Conversely, if μ is a Borel measure on $\mathbb{R}$ that is finite on bounded intervals, and we set

$$F(x)=\begin{cases}\mu((0,x]) & \text{if } 0<x\\ 0 & \text{if } x=0\\ -\mu((x,0]) & \text{if } x<0\end{cases}$$

then, F is increasing and right continuous, and $\mu=\mu_F$ for Borel sets.

Proof: Let $F:\mathbb{R}\to\mathbb{R}$ is increasing and right continuous, and set $\mu_0(\dot{\bigcup}_{j=1}^{n}(a_j,b_j]) = \sum_{j=1}^{n}F(b_j)-F(a_j)$. Then, by Proposition 1.15, we get a premeasure on $\mathcal{A}$, an algebra generated by h-intervals. If we take Lebesgue-Stieltjes outer measure μ_F^* generated by F, then by Caratheodory's Theorem, we know that there exists a measure on a collection of μ_F^*-measurable sets, denoted by $\mathcal{M}_F$. By Proposition 1.13, every set in $\mathcal{A}$ is μ^*-measurable and its measure is μ_0. Since $\mathcal{A}\subseteq\mathcal{M}_F$, σ-algebra generated by $\mathcal{A}$ is a subset of $\mathcal{M}_F$. But, the σ-algebra generated by $\mathcal{A}$ is $\mathcal{B}_{\mathbb{R}}$, the Borel set. Thus, $\mathcal{B}_{\mathbb{R}}\subseteq\mathcal{M}_F$. So, $\mu_F^*|_{\mathcal{B}_{\mathbb{R}}}$ is a measure on $\mathcal{B}_{\mathbb{R}}$. Thus, we have shown that there exists a measure μ_F on $\mathcal{B}_{\mathbb{R}}$ such that $\mu_F((a,b])=\mu_0((a,b])=F(b)-F(a)$.

Done earlier

~~Uniqueness is given by Theorem 1.14~~. Next, if $\mu_F=\mu_G$, then $\mu_F((a,b])=F(b)-F(a)=G(b)-G(a)=\mu_G((a,b])$ for all a and b. Thus, $F(x)=G(x)-G(0)+F(0)$, and so $F-G$ $=$ a constant. Conversely, if $F-G=$ a constant, then $\mu_F((a,b])-\mu_G((a,b])$ for all a and $b\Rightarrow$ give the same premeasure $\Rightarrow$ give the same outer measure, etc.

Suppose that μ is a Borel measure on $\mathbb{R}$ which is finite on bounded intervals, and define

$$F(x)=\begin{cases}\mu((0,x]) & \text{if } 0<x\\ 0 & \text{if } x=0\\ -\mu((x,0]) & \text{if } x<0\end{cases}$$

[Show that F is increasing.]

Let $y\le x$. If $0\le y\le x$, then $F(y)=\mu((0,y])\le\mu((0,x])=F(x)$. If $y\le 0\le x$, then $F(y)\le 0\le F(x)$. If $y\le x\le 0$, then $(y,0]\supseteq(x,0]\Rightarrow\mu((y,0])\ge\mu((x,0])\Rightarrow -\mu((y,0])\le -\mu((x,0])\Rightarrow F(y)\le F(x)$. Thus, F is increasing.

[Show that F is continuous from the right.]
Let $x_n \searrow x$. If $x \geq 0$, then $x_n \geq 0$ and $F(x_n) = \mu((0, x_n])$.
Now, $\lim_{n\to\infty} F(x_n) = \lim_{n\to\infty} \mu((0, x_n]) = \mu(\bigcap_{n=1}^{\infty} (0, x_n]) = \mu((0, x])$
$= F(x)$. If $x < 0$, without loss of generality assume that
$x_n < 0$, then $F(x_n) = -\mu(x_n, 0] \Rightarrow \lim_{n\to\infty} F(x_n) =$

$-\lim_{n\to\infty} \mu((x_n, 0]) = -\mu(\bigcup_{n=1}^{\infty} (x_n, 0]) = -\mu((x, 0]) = F(x)$

because $(x, 0] \supseteq (x_n, 0]$ and $(x, 0] = \bigcup_{n=1}^{\infty} (x_n, 0]$. Thus, F is continuous from the right.

Note that if we require $F(0) = 0$, we can get rid of constant term.
There is a one to one correspondence between "μ is a Borel measure, finite on bounded intervals" and "F is increasing, right continuous and $F(0) = 0$."

Now $\mu((a,b]) = \mu_F((a,b])$ apply HW 7.

Example 1: Fix x_0, and define

$$F(x) = \begin{cases} 0 & \text{if } x < x_0 \\ 1 & \text{if } x_0 \leq x \end{cases}$$

Then, there is a measure μ_F such that

$$\mu_F((a, b]) = \begin{cases} 1 & \text{if } x_0 \in (a, b] \\ 0 & \text{if } x_0 \notin (a, b] \end{cases}$$

$$\mu_F(E) = \begin{cases} 1 & \text{if } x_0 \in E \\ 0 & \text{if } x_0 \notin E \end{cases}$$

$\mu_F \equiv \delta_{x_0}$ is called **dirac delta** or **point mass at x_0**

~~Start with such a Borel measure μ which leads to F, and from F we get an outer measure μ_F^*. Then, by Caratheodory's Theorem, we get $(\mathbb{R}, \mathcal{M}_F, \tilde{\mu}_F)$ which is called **Lebesgue-Stieltjes measure**. Remember that $\tilde{\mu}_F = \mu_F^*|_{\mathcal{M}_F}$.~~

~~The next proposition describe the relationship between $(\mathbb{R}, \mathcal{B}_{\mathbb{R}}, \mu)$ and $(\mathbb{R}, \mathcal{M}_F, \tilde{\mu}_F)$.~~

<u>HW 6.</u> Let F be the c.d.f. for the fair coin discussed earlier. Find μ_F, $\mathcal{M}_F$

~~HW 7. Let $(\mathbb{R}, \mathcal{B}(\mathbb{R}), \mu)$, $(\mathbb{R}, \mathcal{B}(\mathbb{R}), \nu)$ two Borel measures. If $\mu((a,b]) = \nu((a,b])$ $\forall a, b$, then $\mu(B) = \nu(B)$ $\forall B \in \mathcal{B}(\mathbb{R})$~~

<u>HW7</u> Let $(\mathbb{R}, \mathcal{B}(\mathbb{R}), \mu)$, $(\mathbb{R}, \mathcal{B}(\mathbb{R}), \nu)$ be two Borel measures such that

$$\mu((a,b]) = \nu((a,b]) < +\infty, \quad \forall a, b \in \mathbb{R},$$

then $\mu(B) = \nu(B) \quad \forall B \in \mathcal{B}(\mathbb{R})$

<u>Prop'n</u> Let $(\mathbb{R}, \mathcal{B}(\mathbb{R}), \mu)$ be a Borel measure bounded on intervals, F as above. Then $(\mathbb{R}, \mathcal{M}_F, \mu_F)$ is the completion of $(\mathbb{R}, \mathcal{B}(\mathbb{R}), \mu)$

<u>Pf:</u> Let $E \in \mathcal{M}_F$. Assume $\mu_F(E) < +\infty$

Know $\mu_F(E) = \inf \left\{ \sum F(b_n) - F(a_n) \,\middle|\, E \subseteq \bigcup_{n=1}^{\infty} (a_n, b_n] \right.$

Pick $B_k = \bigcup_{n=1}^{\infty} (a_{n,k}, b_{n,k}] \ni$

$$\sum_n F(b_{n,k}) - F(a_{n,k}) \le \mu_F(E) + \frac{1}{k}$$

B_k Borel $\Rightarrow \mu(B_k) = \mu_F(B_k)$. Take $B = \bigcap B_k$

$E \subseteq B \Rightarrow \mu_F(E) \le \mu(B) \le \mu_F(E) + 1/k \quad \forall k$

$\Rightarrow \mu_F(E) = \mu_F(B) \Rightarrow \mu_F(E \setminus B) = 0$

$$E = B \,\dot{\cup}\, \underbrace{(E \setminus B)}_{\text{null set}}$$

General E write $E = \bigcup_{n=-\infty}^{+\infty} (\underbrace{E \cap (n, n+1]}_{E_n})$

Each $E_n = B_n \,\dot{\cup}\, N_n \Rightarrow E = \underbrace{\left(\bigcup B_n\right)}_{\text{Borel}} \dot{\cup} \underbrace{\left(\bigcup N_n\right)}_{\text{null}}$

Multi-dimensions:

Finite unions of $(a,b]\times(c,d]$ algebra of sets A_2 on $\mathbb{R}^2$

If we set $\mu_0\big((a,b]\times(c,d]\big) = (b-a)\cdot(d-c)$ extend to unions, get premeasure on A_2

$(\mathbb{R}^2, \mathcal{M}_2, m_2)$

Resulting measure called 2-dimensional Lebesgue measure m_2. Again A_2 will be contained in $\mathcal{M}_2 \Rightarrow$

$\mathcal{B}(\mathbb{R}^2) \subseteq \mathcal{M}_2$ and $m_2\big((a,b]\times(c,d]\big)$
$= (b-a)(d-c)$

Similarly for k-dimensional Lebesgue measure.

Chapter 2 Integration

2.1 Measurable Functions

Definition: Let $\mathcal{M}$ and $\mathcal{N}$ be σ-algebras. Also, let $(X, \mathcal{M})$ and $(Y, \mathcal{N})$ be measurable spaces. A function $f : X \to Y$ is called **$(\mathcal{M}, \mathcal{N})$-measurable** provided that $f^{-1}(E) \in \mathcal{M}$ for all $E \in \mathcal{N}$.
Recall that $f^{-1}(E) = \{x : f(x) \in E\}$, preimage or inverse image of f.

Proposition: Let $(X, \mathcal{M})$, $(Y, \mathcal{N})$ and $(Z, \mathcal{O})$ be measurable spaces. Let $f : X \to Y$ be $(\mathcal{M}, \mathcal{N})$-measurable and $g : Y \to Z$ be $(\mathcal{N}, \mathcal{O})$-measurable. Then, $g \circ f : X \to Z$ is $(\mathcal{M}, \mathcal{O})$-measurable.

Proof: Let $(X, \mathcal{M})$, $(Y, \mathcal{N})$ and $(Z, \mathcal{O})$ be measurable spaces. Let $f : X \to Y$ be $(\mathcal{M}, \mathcal{N})$-measurable and $g : Y \to Z$ be $(\mathcal{N}, \mathcal{O})$-measurable. Suppose that $E \in \mathcal{O}$. Then, $(g \circ f)^{-1}(E) = \{x : g(f(x)) \in E\} = \{x : f(x) \in g^{-1}(E)\} = f^{-1}(g^{-1}(E))$.
[Note that $g^{-1}(E) \in \mathcal{N}$ and $f^{-1}(g^{-1}(E)) \in \mathcal{M}$.]
Thus, $(g \circ f)^{-1}(E) \in \mathcal{M}$, and so $g \circ f : X \to Z$ is $(\mathcal{M}, \mathcal{O})$-measurable.

Proposition 2.1: Let $(X, \mathcal{M})$ and $(Y, \mathcal{N})$ be measurable spaces, and $\mathcal{N}$ be a σ-algebra generated by $\mathcal{E}$. Let $f : X \to Y$. If $f^{-1}(E) \in \mathcal{M}$ for all $E \in \mathcal{E}$, then f is $(\mathcal{M}, \mathcal{N})$-measurable.

Proof: Let $(X, \mathcal{M})$ and $(Y, \mathcal{N})$ be measurable spaces, and $\mathcal{N}$ be a σ-algebra generated by $\mathcal{E}$. Let $f : X \to Y$. Also, let $f^{-1}(E) \in \mathcal{M}$ for all $E \in \mathcal{E}$, and $\widetilde{\mathcal{N}} = \{B \in Y : f^{-1}(B) \in \mathcal{M}\}$. Then, $\mathcal{E} \subseteq \widetilde{\mathcal{N}}$.
[Note that $\widetilde{\mathcal{N}}$ is a σ-algebra $\Rightarrow \mathcal{N} \subseteq \widetilde{\mathcal{N}} \Rightarrow$ If $B \in \mathcal{N}$, then $f^{-1}(B) \in \mathcal{M}$.]
[Show that $\widetilde{\mathcal{N}}$ is a σ-algebra.]
Since $f^{-1}(\emptyset) = \emptyset \in \mathcal{M}$, $\emptyset \in \widetilde{\mathcal{N}}$. Next, if $B \in \widetilde{\mathcal{N}}$, then $f^{-1}(B) \in \mathcal{M}$. Since $\mathcal{M}$ is a σ-algebra, $(f^{-1}(B))^c = f^{-1}(B^c) \in \mathcal{M}$. Thus, $B^c \in \widetilde{\mathcal{N}}$. Also, if $B_n \in \widetilde{\mathcal{N}}$, then $f^{-1}(B_n) \in \mathcal{M}$. Since $\mathcal{M}$ is a σ-algebra, $\bigcup\limits_n f^{-1}(B_n) = f^{-1}(\bigcup\limits_n B_n) \in \mathcal{M}$, $\bigcup\limits_n B_n \in \widetilde{\mathcal{N}}$. Thus, $\widetilde{\mathcal{N}}$ is a σ-algebra, and so $\mathcal{N} \subseteq \widetilde{\mathcal{N}}$. Therefore, if $B \in \mathcal{N}$, then $B \in \widetilde{\mathcal{N}}$ and so $f^{-1}(B) \in \mathcal{M}$. Thus, f is $(\mathcal{M}, \mathcal{N})$-measurable.