Proposition 1.13: Let $\mathcal{A} \subseteq \mathcal{P}(X)$ be an algebra of sets and μ_0 be a premeasure on $\mathcal{A}$. If μ^* is the outer measure from μ_0 and $\mathcal{M} = \{A : A \text{ is } \mu^*\text{-measurable}\}$, then:

(a) $\mu^*(A) = \mu_0(A)$ for all $A \in \mathcal{A}$

(b) Every $A \in \mathcal{A}$ is μ^*-measurable, that is $\mathcal{A} \subseteq \mathcal{M}$.

Proof of (a): First $A = A \cup \emptyset \cup \emptyset \cup \ldots$, so $\mu^*(A) \leq \mu(A) + \mu(\emptyset) + \mu(\emptyset) + \ldots = \mu(A)$. Thus, $\mu^*(A) \leq \mu(A)$.
[Now, show that it doe not get any smaller.]
Let $A \subseteq \bigcup\limits_{n=1}^{\infty} A_n$ where each $A_n \in \mathcal{A}$. Let $B_n = A \cap (A_n \backslash \bigcup\limits_{j=1}^{n-1} A_j)$.
Then, $B_n \in \mathcal{A}$ and B_ns are all disjoint. Also, notice that $A = \bigcup\limits_{n=1}^{\infty} B_n$. Thus, $\mu(A) = \sum\limits_{n=1}^{\infty} \mu(B_n) \leq \sum\limits_{n=1}^{\infty} \mu(A_n)$, and so $\mu(A) \leq inf\,\{\sum\limits_{n=1}^{\infty} \mu(A_n) : A \subseteq \bigcup\limits_{n=1}^{\infty} A_n \text{ and } A_n \in \mathcal{A}\} \equiv \mu^*(A)$. Thus, $\mu^*(A) = \mu(A)$ for all $A \in \mathcal{A}$.

Proof of (b): Let $A \in \mathcal{A}$ and $E \in X$. (We need to show that $\mu^*(E) \geq \mu^*(E \cap A) + \mu^*(E \cap A^c)$.)
Let $B_j \in \mathcal{A}$ and $\epsilon > 0$ such that $E \subseteq \bigcup\limits_{j=1}^{\infty} B_j$ and $\mu^*(E) \leq \sum\limits_{j=1}^{\infty} \mu(B_j) \leq \mu^*(E) + \epsilon$. Now, $\mu(B_j) = \mu(B_j \cap A) + \mu(B_j \cap A^c)$. So, $\mu^*(E) + \epsilon \geq \sum\limits_{j=1}^{\infty} \mu(B_j) = \sum\limits_{j=1}^{\infty} \mu(B_j \cap A) + \sum\limits_{j=1}^{\infty} \mu(B_j \cap A^c) \geq \mu^*(\bigcup\limits_{j=1}^{\infty} (B_j \cap A)) + \mu^*(\bigcup\limits_{j=1}^{\infty} (B_j \cap A^c)) \geq \mu^*(E \cap A) + \mu^*(E \cap A^c)$. Thus, $\mu^*(E) = \mu^*(E \cap A) + \mu^*(E \cap A^c)$, and so every $A \in \mathcal{A}$ is μ^*-measurable.

Theorem 1.14: Let $\mathcal{A} \subseteq \mathcal{P}(X)$ be an algebra of sets, μ be a premeasure on $\mathcal{A}$ and $\mathcal{M}$ be ~~a~~ the σ-algebra generated by $\mathcal{A}$. Then, $\overline{\mu} = \mu^*|_{\mathcal{M}}$ is a measure on $\mathcal{M}$ such that $\overline{\mu}(A) = \mu(A)$ for all $A \in \mathcal{A}$.
[Uniqueness] If $\nu : \mathcal{M} \to [0, +\infty]$ is a measure such that $\nu(A) = \mu(A)$ for all $A \in \mathcal{A}$, then $\nu(E) \leq \overline{\mu}(E)$ and $\nu(E) = \overline{\mu}(E)$ when $\overline{\mu}(E) < +\infty$ for all $E \in \mathcal{M}$. If μ is σ-finite, then $\nu(E) = \overline{\mu}(E)$ for all $E \in \mathcal{M}$.

Proof: By Proposition 1.13-(b), we know that every set in $\mathcal{A}$ is μ^*-measurable. A collection of μ^*-measurable sets is a σ-algebra by Caratheodory's Theorem.

1.5 Borel Measures on $\mathbb{R}$

Motivation: Suppose that we had a finite measure on the Borel sets of $\mathbb{R}$, denoted by $\mathcal{B}_{\mathbb{R}}$. Say, $\mu : \mathcal{B}_{\mathbb{R}} \to [0, +\infty]$ is a finite measure.
Let $F(x) = \mu((-\infty, x])$ which is called the **(cumulative) distribution function of μ.**
If $x_n \searrow x$, then $\lim_{n\to\infty} F(x_n) = \lim_{n\to\infty} \mu((-\infty, x_n]) = \mu(\bigcap_{n=1}^{\infty} (-\infty, x_n]) =$
$\mu(-\infty, x] = F(x)$. Thus, F is continuous from the right at x.
If $x_n \nearrow x$, then $\bigcup_{n=1}^{\infty} (-\infty, x_n] = (-\infty, x)$. So, $\lim_{n\to\infty} F(x_n) =$
$\lim_{n\to\infty} \mu((-\infty, x_n]) = \mu(\bigcup_{n=1}^{\infty} (-\infty, x_n]) = \mu((-\infty, x)) \leq \mu((-\infty, x]) = F(x)$
Now, $(-\infty, x] = (-\infty, x) \cup \{x\}$, so $\mu((-\infty, x]) = \mu((-\infty, x)) \cup \mu(\{x\})$.
Thus, if $\mu(\{x\}) \neq 0$, then F is not continuous from the left at x.
So, when we do Lebesgue-Stieltjes outer measure, we will really only look at F right continuous.

Let $F : \mathbb{R} \to \mathbb{R}$ be increasing and right continuous. Let $\mathcal{A}$ be an algebra generated by sets of the form $(a, b]$, $(-\infty, b]$ and $(a, +\infty)$. (Note that they are called **h-intervals.**) Set $F(+\infty) = \lim_{x\to+\infty} F(x)$ and $F(-\infty) = \lim_{x\to-\infty} F(x)$.
Now, for any h-interval, set $\mu_0((a, b]) = F(b) - F(a)$, $\mu_0((a, +\infty)) = F(+\infty) - F(a)$ and $\mu_0((-\infty, b]) = F(b) - F(-\infty)$.

Proposition 1.15: Let F be right continuous and increasing. For $\overset{A=}{\bigcup_{j=1}^{n} (a_j, b_j]}$ $\in \mathcal{A}$, algebra, which are all disjoint, set $\mu_0(\bigcup_{j=1}^{n} (a_j, b_j]) = \sum_{j=1}^{n} F(b_j) - F(a_j)$.
Then, μ_0 is well-defined, and is a premeasure on $\mathcal{A}$.

Proof: [Show that μ_0 is well-defined.]

First suppose that $(a, b] = \bigcup_{j=1}^{n} (a_j, b_j]$ where each $(a_j, b_j]$ is

$a_1 < a_2 < \ldots < a_m$

disjoint. After relabeling $(a_j, b_j]$ as $a = a_1$, $b_1 = a_2$, $b_2 = a_3$, $\ldots, b_{n-1} = a_n, b_n = b$, $\mu_0(\bigcup_{j=1}^{n} (a_j, b_j]) = \sum_{j=1}^{n} F(b_j) - F(a_j) =$
$F(b_1) - F(a_1) + F(b_2) - F(a_2) + \ldots + F(b_n) - F(a_n) =$
$F(b_n) - F(a_1) = F(b) - F(a)$.

Next, let $A = \dot{\bigcup}_{j=1}^{n} (a_j, b_j] = \dot{\bigcup}_{i=1}^{m} (c_i, d_i]$. Let $J_j = (a_j, b_j]$ and $I_i = (c_i, d_i]$, then $\{J_j \cap I_i : i, j\}$ is a collection of disjoint sets.
Now, $J_j = \dot{\bigcup}_{i=1}^{m} (J_j \cap I_i)$, and let $J_j \cap I_i = (e_{ij}, f_{ij}]$. By the first case, $F(b_j) - F(a_j) = \mu_0(\dot{\bigcup}_{i=1}^{m} (J_j \cap I_i)) = \sum_{i=1}^{m} F(f_{ij}) - F(e_{ij})$.
Thus, $\sum_{j=1}^{n} F(b_j) - F(a_j) = \sum_{j=1}^{n} \sum_{i=1}^{m} F(f_{ij}) - F(e_{ij}) = \sum_{i=1}^{m} F(d_i) - F(c_i)$. Thus, μ_0 is well-defined.
[Show that μ_0 is a premeasure on $\mathcal{A}$.]
<u>Case</u>: $(a, b] = \dot{\bigcup}_{n=1}^{\infty} (a_n, b_n]$
Since $b \in (a, b]$, there exists i_0 such that $b \in (a_{i_0}, b_{i_0}]$, that is $b = b_{i_0}$. Relabel i_0 with 1 so that $b_1 = b_{i_0} = b$, $a_1 = a_{i_0}$ and $a \le a_1$. Now, $a_1 \in (a, b]$ implies that there exists i_1 such that $a_1 \in (a_{i_1}, b_{i_1}]$, that is $b_{i_1} = a_1$. Relabel $b_{i_1} = b_2$, $a_{i_1} = a_2$ and $(a_2, b_2]$ with $b_2 = a_1$. Continue relabeling with this manner, we see that $(a, b] = \dot{\bigcup}_{n=1}^{\infty} (a_n, b_n]$ where $b_1 = b$, $b_2 = a_1, \ldots,$ $b_n = a_{n-1}$ and $a_1 > a_2 > \ldots > a$ with $\lim_{n\to\infty} a_n = a$.
(We need to show that $\mu_0((a, b]) = \sum_{n=1}^{\infty} \mu_0((a_n, b_n])$.)
The R.H.S. $= \sum_{n=1}^{\infty} \mu_0((a_n, b_n]) = \lim_{n\to\infty} \sum_{j=1}^{n} \mu_0((a_j, b_j]) = \lim_{n\to\infty} [F(b_1) - F(a_1) + F(b_2) - F(a_2) + \ldots + F(b_n) - F(a_n)] = \lim_{n\to\infty} [F(b_1) - F(a_n)] = \lim_{n\to\infty} [F(b) - F(a_n)] = F(b) - F(a) = \mu_0((a, b])$. Thus, μ_0 is a premeasure on $\mathcal{A}$.
<u>General case</u>: Use case $(a, b] = \dot{\bigcup}_{n=1}^{\infty} (a_n, b_n]$, double indexing and the similar argument.

<u>Theorem</u>: Let $F : \mathbb{R} \to \mathbb{R}$ be increasing and right continuous. Then there exists a Borel measure $\mu_F : \mathcal{B}(\mathbb{R}) \to [0, +\infty]$ such that $\mu_F((a, b]) = F(b) - F(a)$

<u>Pf</u>: By Caratheodory the outer measure μ_0^* where $\mu_0((a, b]) = F(b) - F(a)$ gives rise to a measure on the μ_0^*-measurable sets, $\mathcal{M}_0$. Since μ_0 is a premeasure on $\mathcal{A}$ we have that $\mathcal{A} \subseteq \mathcal{M}$ and so the σ-algebra generated by $\mathcal{A}$ is in $\mathcal{M}$. But the σ-algebra generated by $\mathcal{A}$ is $\mathcal{B}(\mathbb{R})$

So, μ_F is just μ_0^* restricted to the Borel sets

Finally, $\mu_F((a, b]) = \mu_0^*((a, b]) = \mu_0((a, b]) = F(b) - F(a)$, since for premeasures we know $\mu_0^*(A) = \mu_0(A)$ when $A \in \mathcal{A}$.