1/16 - 1

One more example of outer measure

<u>Lebesgue-Stieltjes</u>: $F: \mathbb{R} \to \mathbb{R}$ increasing and continuous from right

$$\mu_F^*(E) = \inf\left\{ \sum_{1}^{\infty} F(b_i) - F(a_i) \;\middle|\; E \subseteq \bigcup_{i=1}^{\infty} (a_i, b_i] \right\}$$

$F(t) = t$ get Lebesgue outer measure

Comes from <u>cumulative</u> distribution functions c.d.f.

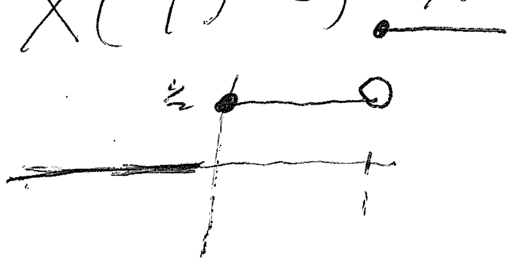
Given $(\Omega, \mathcal{M}, \mu)$ - probability space

$X: \Omega \to \mathbb{R}$. The cdf of X is

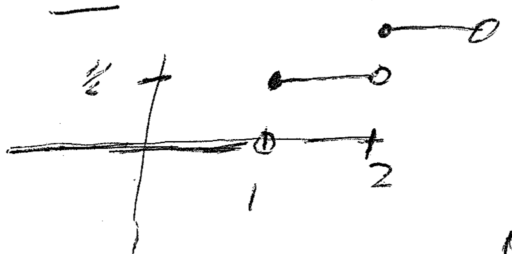
$F_X(t) = \mu(\{\omega \in \Omega : X(\omega) \leq t\})$ increasing right continuous

<u>Ex</u> Fair coin $\Omega = \{H, T\}$, $\mu(\{H\}) = \mu(\{T\}) = 1/2$

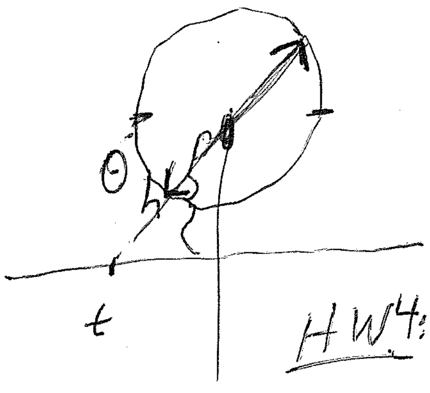
$X(T) = 0$, $X(H) = 1$



<u>Ex</u>. Fair die $\Omega = \{1, 2, 3, 4, 5, 6\}$, $\mu(\{i\}) = 1/6$



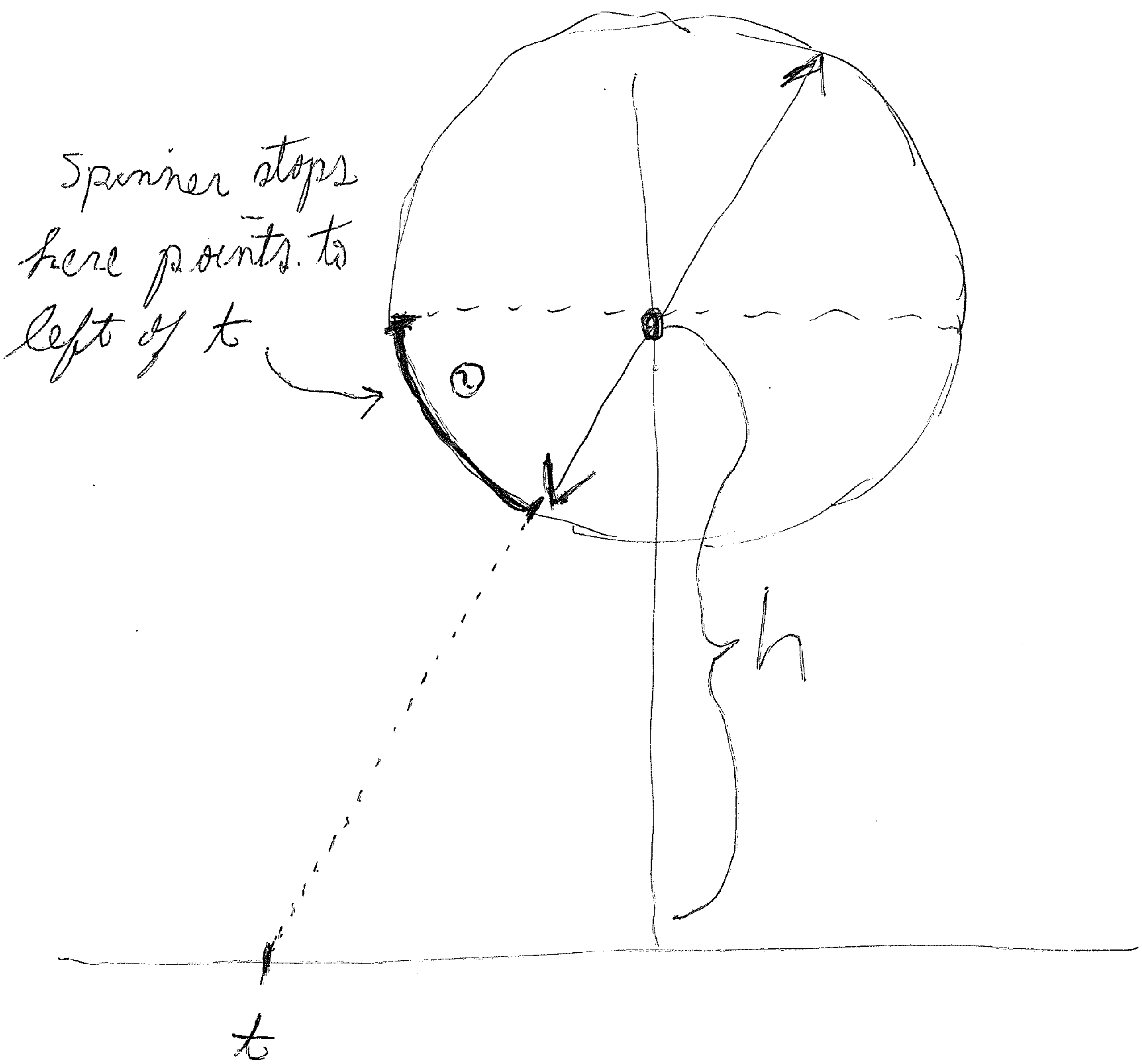
<u>Ex</u> Spinner



$F_X(t)$ = prob. spinner points to the right of $t = \frac{\theta}{\pi}$

<u>HW4</u>: Find $F_X(t)$ as a function of t

1/16 - 1.1 Blown Up Picture



$$F_X(0) = \frac{\pi/2}{\pi} = 1/2$$

This is an example of a "continuous probability space".

$A, B \in \mathcal{M} \Rightarrow A \cup B$

Next, let $A, B \in \mathcal{M}$ and $E \subseteq X$. Then, $\mu^*(E) = \mu^*(E \cap A) + \mu^*(E \cap A^c) = \mu^*((E \cap A) \cap B) + \mu^*((E \cap A) \cap B^c) + \mu^*((E \cap A^c) \cap B) + \mu^*((E \cap A^c) \cap B^c)$.

Note that $E \cap (A \cup B) = (E \cap A \cap B) \dot{\cup} (E \cap A \cap B^c) \dot{\cup} (E \cap A^c \cap B)$, and $E \cap (A \cup B)^c = E \cap A^c \cap B^c$.

So, $\mu^*(E) = \mu^*(E \cap A) + \mu^*(E \cap A^c) = \mu^*((E \cap A) \cap B) + \mu^*((E \cap A) \cap B^c) + \mu^*((E \cap A^c) \cap B) + \mu^*((E \cap A^c) \cap B^c) \geq \mu^*(E \cap (A \cup B)) + \mu^*(E \cap (A \cup B)^c)$. Thus, $\mu^*(E) = \mu^*(E \cap (A \cup B)) + \mu^*(E \cap (A \cup B)^c)$, and so $A \cup B \in \mathcal{M}$.

Hence, $\mathcal{M}$ algebra

Inductively, $A_1, A_2, \ldots, A_n \in \mathcal{M} \Rightarrow A_1 \cup A_2 \cup \ldots \cup A_n \in \mathcal{M}$.

[~~By homework problem,~~ It is enough to show that if $\{A_n\} \subseteq \mathcal{M}$ and A_ns are disjoint, then $B = \bigcup_{n=1}^{\infty} A_n \in \mathcal{M}$.] Since $A_1 \cup A_2 = A_1 \cup (A_2 \setminus A_1)$

HW5: If $\mathcal{M}$ is an algebra of sets, and whenever $\{A_n\}$ disjoint $\bigcup_{n=1}^{\infty} A_n \in \mathcal{M}$ Then $\mathcal{M}$ σ-algebra

Let $B_m = \bigcup_{n=1}^{m} A_n \in \mathcal{M}$.

<u>Lemma</u>: For any E, $\mu^*(E \cap B_m) = \mu^*(E \cap A_1) + \mu^*(E \cap A_2) + \ldots + \mu^*(E \cap A_m)$.

Proof of Lemma: Let $A_m \in \mathcal{M}$. Then, $\mu^*(E \cap B_m) = \mu^*(E \cap B_m \cap A_m) + \mu^*(E \cap B_m \cap A_m^c) = \mu^*(E \cap A_m) + \mu*(E \cap B_{m-1})$ (So, inductively) $= \mu^*(E \cap A_m) + \mu^*(E \cap A_{m-1}) + \ldots + \mu^*(E \cap A_1)$.

Now, since $B_m \in \mathcal{M}$, for any E, $\mu^*(E) = \mu^*(E \cap B_m) + \mu^*(E \cap B_m^c) = \sum_{n=1}^{m} \mu^*(E \cap A_n) + \mu^*(E \cap B_m^c) \geq \sum_{n=1}^{m} \mu^*(E \cap A_n) + \mu^*(E \cap B^c)$ for all m. Thus, $\mu^*(E) = \sum_{n=1}^{\infty} \mu^*(E \cap A_n) + \mu^*(E \cap B^c) \geq \mu^*(\bigcup_{n=1}^{\infty} (E \cap A_n)) + \mu^*(E \cap B^c) = \mu^*(E \cap B) + \mu^*(E \cap B^c)$. Thus, $\mu^*(E) = \mu^*(E \cap B) + \mu^*(E \cap B^c)$, and so $B \in \mathcal{M}$.

Thus, $\mathcal{M}$ is a σ-algebra.

[Next, show that μ is a measure.]

First, $\mu(\emptyset) = \mu^*(\emptyset) = 0$.

Next, let $A_n \in \mathcal{M}$ be disjoint and $B = \bigcup_{n=1}^{\infty} A_n$.

(We need to show that $\mu(B) = \sum_{n=1}^{\infty} \mu(A_n)$.)

Use the above by letting $E = B$, $\mu(B) = \mu^*(B) \geq$

$\sum_{n=1}^{\infty} \mu^*(B \cap A_n) + \mu^*(B \cap B^c) = \sum_{n=1}^{\infty} \mu(A_n) + \mu^*(\emptyset) =$
$\sum_{n=1}^{\infty} \mu(A_n) + 0 = \sum_{n=1}^{\infty} \mu(A_n)$. Thus, $\mu(B) \geq \sum_{n=1}^{\infty} \mu(A_n)$, and so $\mu(B) = \sum_{n=1}^{\infty} \mu(A_n)$. Therefore, μ is a measure.

[Show that μ is a complete measure.]
Let $A \in \mathcal{M}$ and $\mu(A) = \mu^*(A) = 0$. Then, $A \in \mathcal{N}$. Let $B \subseteq A$. (We need to show that $B \in \mathcal{M}$.)
Since $B \subseteq A, 0 < \mu^*(B) \leq \mu^*(A) = 0$. So, $\mu^*(B) = 0$ and $\mu^*(E \cap B) = 0$. Thus, $\mu^*(E \cap B) + \mu^*(E \cap B^c) = 0 + \mu^*(E \cap B^c) \leq \mu^*(E)$, and so $\mu^*(E) = \mu^*(E \cap B) + \mu^*(E \cap B^c)$. Therefore, $B \in \mathcal{M}$, and thus μ is a complete measure.

Recall that given $\mathbb{R}$ and letting m^* be Lebesgue outer measure, if we know that $\mathcal{M} = \{A : A \text{ is a } m^*\text{-measurable set}\}$, then $\mathcal{M}$ is a σ-algebra and $m^*|_{\mathcal{M}} = m$ is a measure. But, we don't know that if $(a, b) \in \mathcal{M}$, ~~then~~ $m^*((a, b)) = b - a$. ~~(Proposition 1.13 takes care of this problem.)~~

Definition: Let $\mathcal{A} \subseteq \mathcal{P}(X)$ be an algebra of sets. Then, $\mu_0 : \mathcal{A} \to [0, +\infty]$ is called a **premeasure** on $\mathcal{A}$ provided that:

(i) $\mu_0(\emptyset) = 0$

(ii) If $\{A_n\} \subseteq \mathcal{A}$, A_ns are disjoint and $\bigcup_{n=1}^{\infty} A_n \in \mathcal{A}$, then $\mu_0(\bigcup_{n=1}^{\infty} A_n) = \sum_{n=1}^{\infty} \mu_0(A_n)$.

Note that given a premeasure μ_0 and if we set $\mu^*(E) = inf\,\{\sum_{n=1}^{\infty} \mu_0(A_n) : E \subseteq \bigcup_{n=1}^{\infty} A_n \text{ and } A_n \in \mathcal{A}\}$, then μ^* is an outer measure.

The following proposition takes care of the problem "we don't know that if $(a, b) \in \mathcal{M}$, then $m^*((a, b)) = b - a$."