Observe that:

(1) μ is finite if $\mu(E) < +\infty$ for all $E \in \mathcal{M}$ if and only if μ is finite if $\mu(X) < +\infty$.
(2) If μ is finite, then μ is σ-finite.
(3) If μ is σ-finite, then μ is semifinite.

Proof (3): Suppose that μ is σ-finite. Then, $X = \bigcup\limits_{n=1}^{\infty} X_n$, $\mu(X_n) < +\infty$ and $X_n \in \mathcal{M}$ for all n. Let $E \in \mathcal{M}$ with $\mu(E) = +\infty$. Suppose that $E_n = E \cap (\bigcup\limits_{j=1}^{n} X_j)$. Then, $E_1 \subseteq E_2 \subseteq \ldots$ and $\bigcup\limits_{n=1}^{\infty} E_n = E$. Now, $\mu(E_n) \leq \mu(\bigcup\limits_{j=1}^{n} X_j)$ $= \sum\limits_{j=1}^{n} \mu(X_j) < +\infty$. Since $\lim\limits_{n\to\infty} \mu(E_n) = \mu(\bigcup\limits_{n=1}^{\infty} E_n) =$ $\mu(E) = +\infty$, there exists n such that $\mu(E_n) > 0$.

Definition: Let $(X, \mathcal{M}, \mu)$ be a measure space. If $N \in \mathcal{M}$ and $\mu(N) = 0$, then N is called a **null set (μ-null set)**. We let $\mathcal{N} = \{N \in \mathcal{M} : \mu(N) = 0\}$ be the collection of μ-null sets.

Note that: (1) If $N_j \in \mathcal{N}$, then $\bigcup\limits_{j=1}^{\infty} N_j \in \mathcal{N}$.

(2) If $N \in \mathcal{N}$, $E \in \mathcal{M}$ and $E \subseteq N$, then $\mu(E) = 0$ by monotonicity. (In general, it need not be true that $E \in \mathcal{M}$)

Definition: We call $(X, \mathcal{M}, \mu)$ a **complete measure space** provided that if $N \in \mathcal{N}$ and $E \subseteq N$, then $E \in \mathcal{M}$.

Theorem 1.9: Let $(X, \mathcal{M}, \mu)$ be a measure space, $\mathcal{N}$ be ~~a~~ the collection of null sets, and $\overline{\mathcal{M}} = \{E \cup F : E \in \mathcal{M}, F \subseteq N, N \in \mathcal{N}\}$. Then, $\overline{\mathcal{M}}$ is a σ-algebra, and if we define $\overline{\mu} : \overline{\mathcal{M}} \to [0, +\infty]$ by $\overline{\mu}(E \cup F) = \mu(E)$, then $\overline{\mu}$ is well-defined and is a measure on $\overline{\mathcal{M}}$. Moreover, if $\nu : \overline{\mathcal{M}} \to [0, +\infty]$ is any measure on $\overline{\mathcal{M}}$ such that $\nu(E) = \mu(E)$ for all $E \in \mathcal{M}$, then $\nu = \overline{\mu}$.

Proof: [Show that $\overline{\mathcal{M}}$ is a σ-algebra.]
Let $\mathcal{M}$ be a σ-algebra and $\overline{\mathcal{M}}$ be as stated above. Suppose that $E \cup F \in \overline{\mathcal{M}}$. Then, $(E \cup F)^c = (E \cup N)^c \cup (N \backslash (E \cup F))$. Since $E, N \in \mathcal{M} \Rightarrow E \cup N \in \mathcal{M} \Rightarrow (E \cup N)^c \in \mathcal{M}$ and $N \backslash (E \cup F) \subseteq N \in \mathcal{N}$, $(E \cup F)^c \in \overline{\mathcal{M}}$.

Next, let $E_n \cup F_n \in \overline{\mathcal{M}}$ where $E_n \in \mathcal{M}$, $F_n \subseteq N_n$ and $N_n \in \mathcal{N}$ for all n. Then, $\bigcup_{n=1}^{\infty}(E_n \cup F_n) = (\bigcup_{n=1}^{\infty} E_n) \cup (\bigcup_{n=1}^{\infty} F_n)$. Notice that $\bigcup_{n=1}^{\infty} E_n \in \mathcal{M}$ and $\bigcup_{n=1}^{\infty} F_n \subseteq \bigcup_{n=1}^{\infty} N_n \in \mathcal{N}$, and so $\bigcup_{n=1}^{\infty}(E_n \cup F_n) \in \overline{\mathcal{M}}$. Thus, $\overline{\mathcal{M}}$ is a σ-algebra.

[Show that $\overline{\mu}$ is well-defined.]

Suppose that $E_1 \cup F_1 = E_2 \cup F_2$, $F_1 \subseteq N_1$ and $F_2 \subseteq N_2$. (We need to show that $\mu(E_1) = \mu(E_2)$.) Then, $E_1 \subseteq E_1 \cup F_1 \subseteq E_2 \cup F_2 \subseteq E_2 \cup N_2$. This implies that $\mu(E_1) \leq \mu(E_2 \cup N_2) \leq \mu(E_2) + \mu(N_2) = \mu(E_2) + 0 = \mu(E_2)$. Similarly, $\mu(E_2) \leq \mu(E_1)$, and so $\mu(E_1) = \mu(E_2)$. Thus, $\overline{\mu}$ is well-defined.

[Show that $\overline{\mu}$ is a measure on $\overline{\mathcal{M}}$.]

It is clear that $\overline{\mu}(\emptyset) = 0$. Next, let $\{E_n \cup F_n\}$ be disjoint where $E_n \in \mathcal{M}$, $F_n \subseteq N_n$ and $N_n \in \mathcal{N}$ for all n If $\bigcup_{n=1}^{\infty}(E_n \cup F_n) = (\bigcup_{n=1}^{\infty} E_n) \cup (\bigcup_{n=1}^{\infty} F_n)$, then $\overline{\mu}(\bigcup_{n=1}^{\infty}(E_n \cup F_n)) = \mu((\bigcup_{n=1}^{\infty} E_n)) = \sum_{n=1}^{\infty}\mu(E_n) = \sum_{n=1}^{\infty}\overline{\mu}(E_n \cup F_n)$. Thus, $\overline{\mu}$ is a measure on $\overline{\mathcal{M}}$.

<u>Notation</u>: $E \triangle F = (E \backslash F) \cup (F \backslash E) = (E \cup F) \backslash (E \cap F)$ is called the **symmetric difference.**

In general it is hard to construct Borel measures. We will do it in several steps: ① Build "outer" measures defined ∀ sets
② Restrict to a "special" family of subsets
③ Show that this special family contains open sets.

1.4 Outer Measures

Outer Measure is a key way to construct measures.

Definition: Let X be a nonempty set. A function $\mu^* : \mathcal{P}(X) \to [0, +\infty]$ is called an **outer measure** if:

(i) $\mu^*(\emptyset) = 0$

(ii) (Monotonicity) If $A \subseteq B$, then $\mu^*(A) \leq \mu^*(B)$.

(iii) (Countable Subadditivity) $\mu^*(\bigcup\limits_{n=1}^{\infty} A_n) \leq \sum\limits_{n=1}^{\infty} \mu^*(A_n)$

Example 1: Lebesgue's Outer Measure on $\mathbb{R}$

Given $E \subseteq \mathbb{R}$, $m^*(E) = inf\, \{\sum\limits_{n=1}^{\infty} (b_n - a_n) : E \subseteq \bigcup\limits_{n=1}^{\infty} (a_n, b_n)\}$

Proposition: m^* is an outer measure on $\mathbb{R}$.

Proof: It is clear that $m^*(\emptyset) - 0$.

If $A \subseteq B$ and $B \subseteq \bigcup\limits_{n=1}^{\infty} (a_n, b_n)$, then $A \subseteq \bigcup\limits_{n=1}^{\infty} (a_n, b_n)$. So, inf for A is smaller. Thus, $m^*(A) \leq m^*(B)$.

Next, let $E = \bigcup\limits_{i=1}^{\infty} E_i$. [Show that $m^*(E) \leq \sum\limits_{i=1}^{\infty} m^*(E_i)$]

If $m^*(E_i) = +\infty$ for any i, then the above is true. Assume that $m^*(E_i) < +\infty$ for all i. Fix $\epsilon > 0$, then we can choose (a_n^i, b_n^i) such that $E_i \subseteq \bigcup\limits_{n=1}^{\infty} (a_n^i, b_n^i)$ and $\sum\limits_{n=1}^{\infty} (b_n^i - a_n^i) \leq m^*(E_i) + \epsilon/2^i$.

Then, $E \subseteq \bigcup\limits_{i=1}^{\infty} \bigcup\limits_{n=1}^{\infty} (a_n^i, b_n^i)$. So, $m^*(E) \leq \sum\limits_{i=1}^{\infty} \sum\limits_{n=1}^{\infty} (b_n^i - a_n^i) \leq \sum\limits_{i-1}^{\infty} (m^*(E_i) + \epsilon/2^i) = (\sum\limits_{i-1}^{\infty} m^*(E_i)) + \epsilon$. This implies that $m^*(E) \leq \sum\limits_{i=1}^{\infty} m^*(E_i)$. Thus, m^* is an outer measure on $\mathbb{R}$.

<u>Note</u> that there is nothing special about open intervals.

Suppose that we define $\tilde{m}^*(E) = inf\, \{\sum\limits_{n=1}^{\infty} (b_n - a_n) : E \subseteq \bigcup\limits_{n=1}^{\infty} [a_n, b_n]\}$

Then, $\tilde{m}^*(E) \leq m^*(E)$.

Given $\epsilon > 0$, we can pick $E \subseteq \bigcup_{n=1}^{\infty} [a_n, b_n]$ such that $\sum_{n=1}^{\infty} (b_n - a_n) \leq$ $\widetilde{m}^*(E) + \epsilon \Rightarrow E \subseteq \bigcup_{n=1}^{\infty} (a_n - \epsilon/2^n, b_n + \epsilon/2^n) \Rightarrow m^*(E) \leq$ $\sum_{n=1}^{\infty} (b_n + \epsilon/2^n) - (a_n - \epsilon/2^n) = \sum_{n=1}^{\infty} (b_n - a_n + 2 \cdot \epsilon/2^n) =$ $\sum_{n=1}^{\infty} (b_n - a_n) + 2\epsilon \leq \widetilde{m}^*(E) + 3\epsilon$. This is true for all $\epsilon > 0 \Rightarrow$ $m^*(E) \leq \widetilde{m}^*(E)$ and so $m^*(E) = \widetilde{m}^*(E)$.

Similarly, $m^*(E) = inf\,\{\sum_{n=1}^{\infty} (b_n - a_n) : E \subseteq \bigcup_{n=1}^{\infty} (a_n, b_n]\}$.

Example 2: Lebesgue-Stieltjes Outer Measure

Let $F : \mathbb{R} \rightarrow \mathbb{R}$ be increasing, that is $a \leq b \Rightarrow F(a) \leq F(b)$. and F continuous ~~and Borel~~ from the right

Define $\mu_F^*(E) = inf\,\{\sum_{n=1}^{\infty} F(b_n) - F(a_n) : E \subseteq \bigcup_{n=1}^{\infty} (a_n, b_n]\}$.

Example 3: Lebesgue's Planar Outer Measure on $\mathbb{R}^2$

Let $R = (a, b) \times (c, d) = \{(x, y) : a < x < b, c < y < d\}$, and $A(R) = (b - a)(d - c)$.

Given $E \subseteq \mathbb{R}^2$, define $m_2^*(E) = inf\,\{\sum_{n=1}^{\infty} A(R_n) : E \subseteq \bigcup_{n=1}^{\infty} R_n\}$ where R_ns are rectangles.

Example 4: Lebesgue's Outer Measure on $\mathbb{R}^3$

Let $B = (a, b) \times (c, d) \times (e, f)$ and $V\mathcal{L}(B) = (b - a)(d - c)(f - e)$

Given $E \subseteq \mathbb{R}^3$, define $m_3^*(E) = inf\,\{\sum_{n=1}^{\infty} V\mathcal{L}(B_n) : E \subseteq \bigcup_{n=1}^{\infty} B_n\}$ where B_ns are boxes.

<u>Note</u> that definition of m_k^* on $\mathbb{R}^k$ is similar to the above.

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Proposition 1.10: Let X be a nonempty set. Let $\mathcal{E} \subseteq \mathcal{P}(X)$ such that $\emptyset \in \mathcal{E}$ and there exists $E_n \in \mathcal{E}$ such that $X \subseteq \bigcup_{n=1}^{\infty} E_n$. Let $\rho : \mathcal{E} \to [0, +\infty]$ with $\rho(\emptyset) = 0$. If we set $\mu^*(A) = inf\,\{\sum_{n=1}^{\infty} \rho(E_n) : A \subseteq \bigcup_{n=1}^{\infty} E_n\}$, then μ^* is an outer measure on X.

Proof: It is clear that $\mu^*(\emptyset) = \rho(\emptyset) = 0$, and if $A \subseteq B$, then $\mu^*(A) \leq \mu^*(B)$.

Next, let $A = \bigcup_{i=1}^{\infty} A_i$. [Show that $\mu^*(A) \leq \sum_{i=1}^{\infty} \mu^*(A_i)$.]

If $\mu^*(A_i) = +\infty$ for any i, then we are done.

Assume that $\mu^*(A_i) < +\infty$ for all i. Fix $\epsilon > 0$ and pick $E_{n,i} \in \mathcal{E}$ such that $A_i \subseteq \bigcup_{n=1}^{\infty} E_{n,i}$ and $\sum_{n=1}^{\infty} \mu^*(E_{n,i}) \leq \mu^*(A_i) + \epsilon/2^i$. Then, $A \subseteq \bigcup_{i=1}^{\infty} \bigcup_{n=1}^{\infty} E_{n,i} \Rightarrow \mu^*(A) \leq \sum_{i=1}^{\infty} \sum_{n=1}^{\infty} \mu^*(E_{n,i}) \leq \sum_{i=1}^{\infty} \mu^*(A_i) + \epsilon \Rightarrow \mu^*(A) \leq \sum_{i=1}^{\infty} \mu^*(A_i)$.

Thus, μ^* is an outer measure on X.

End here

Note that by this proposition, Examples 2, 3, and 4 are all outer measures.

Definition: Let μ^* be an outer measure on X. A set $A \subseteq X$ is called **μ^*-measurable** provided that for all $E \subseteq X$, $\mu^*(E) = \mu^*(E \cap A) + \mu^*(E \cap A^c)$.

Note that $\mu^*(E) \leq \mu^*(E \cap A) + \mu^*(E \cap A^c)$ is always true.

Caratheodory's Theorem: If μ^* is an outer measure on X and $\mathcal{M} = \{A : A \text{ is } \mu^*\text{-measurable}\}$, then $\mathcal{M}$ is a σ-algebra, and setting $\mu : \mathcal{M} \to [0, +\infty]$ by $\mu(A) = \mu^*(A)$ is a complete measure on $\mathcal{M}$.

Note that $\mu = \mu^*|_{\mathcal{M}}$

Proof: [Show that $\mathcal{M}$ is a σ-algebra.]

It is clear that $\emptyset \in \mathcal{M}$ because for all $E \subseteq X$, $\mu^*(E \cap \emptyset) = \mu^*(\emptyset) = 0$ and $\mu^*(E \cap \emptyset^c) = \mu^*(E \cap X) = \mu^*(E)$. Thus, $\mu^*(E) = \mu^*(E \cap \emptyset) + \mu^*(E \cap \emptyset^c)$. Thus, $\mathcal{M} \neq \emptyset$.

Suppose that $A \in \mathcal{M}$. Then, $\mu^*(E) = \mu^*(E \cap A) + \mu^*(E \cap A^c) = \mu^*(E \cap A^c) + \mu^*(E \cap (A^c)^c)$ for all $E \subseteq X$.

Thus, $A^c \in \mathcal{M}$.