1.4 $L^p - L^q$ Duality $(1 \le p, q \le +\infty, 1/p + 1/q = 1)$

Definition: A linear map $T : L^p(X, \mathfrak{A}, \mu) \to \mathbb{R}$ is called a **linear functional.** If there exists $M \ge 0$ such that $|T(f)| \le M||f||_p$ for all $[f] \in L^p$, then T is called a **bounded linear functional** (b.l.f.). The least such M is called the **norm of T**, denoted by $||T||$.

Proposition: Let $g \in \mathcal{L}^q$. Then setting $T_g : L^p(X, \mathfrak{A}, \mu) \to \mathbb{R}$ via $T_g([f]) = \int_X fgd\mu$ defines a b.l.f. with $||T_g|| = ||g||_q$. Moreover, $T_{g_1} = T_{g_2}$ if and only if $g_1 = g_2\, a.e.\mu$ (That is $[g_1] = [g_2]$)

Proof: By Holder's inequality, if $f \in \mathcal{L}^p$, then $\left|\int_X fgd\mu\right| \le ||f||_p||g||_q$. So, $\int_X fgd\mu$ is a finite number for all $f \in \mathcal{L}^p$. If $[f_1] = [f_2] \Rightarrow$ $f_1 - f_2 = 0\, a.e.\mu \Rightarrow \int_X (f_1 - f_2)gd\mu = 0 \Rightarrow \int_X f_1gd\mu = \int_X f_2gd\mu$
This shows that T_g is a well-defined map.
Check for T_g Linear: Let $[f_1], [f_2] \in L^p \Rightarrow [f_1] + [f_2] = [f_1 + f_2]$. So, $T_g([f_1 + f_2]) = \int_X (f_1 + f_2)gd\mu = \int_X f_1gd\mu + \int_X f_2gd\mu =$ $T_g([f_1]) + T_g([f_2])$. Similarly, $T_g(\alpha[f]) = \alpha T_g([f])$.
Check for T_g bounded: Apply Holder, $|T_g([f])| = \left|\int_X fgd\mu\right| \le$ $||f||_p||g||_q$. Thus, T_g is bounded by $||g||_q \Rightarrow ||T_g|| \le ||g||_q$.
Case $1 < p < +\infty$ and $1 < q < +\infty$: Let $f(x) =$ $sgn(g(x))|g(x)|^{q-1}$. Then, $f(x)g(x) = sgn(g(x))g(x)|g(x)|^{q-1} =$ $|g(x)|^q$. Thus, $||g||_q^q = \int_X |g|^q d\mu = T_g(f) \le ||T_g||||f||_p =$ $||T_g||(\int_X |f(x)|^p d\mu)^{1/p} = ||T_g||\left(\int_X |g(x)|^{(q-1)p} d\mu\right)^{1/p} =$ $||T_g||(\int_X |g(x)|^q d\mu)^{1/p} = ||T_g||||g||_q^{q/p} \Rightarrow ||g||_p^{q-q/p} \le ||T_g|| \Rightarrow$ $|g||_q \le ||T_g||$, and so $||g||_q = ||T_g||$.
Finally, if $g_1 = g_2\, a.e.\mu$, then $||g_1 - g_2|| = 0 \to ||T_{(g_1-g_2)}|| = 0 \Rightarrow$ $T_{(g_1-g_2)} = 0$ for all f. $\int_X f(g_1 - g_2)d\mu = 0$, $\int_X fg_1d\mu = \int_X fg_2d\mu$, $T_{g_1(f)} = T_{g_2(f)}$ for all f. So, $g_1 = g_2\, a.e.\mu \Rightarrow T_{g_1} = T_{g_2}$.
But, $T_{g_1} = T_{g_2} \Rightarrow ||T_{g_1} - T_{g_2}|| = 0 \Rightarrow ||T_{g_1} - T_{g_2}|| = ||g_1 - g_2||_q = 0$ $\Rightarrow g_1 = g_2\, a.e.\mu$.

Proposition: Let $T : L^p(X, \mathfrak{A}, \mu) \to \mathbb{R}$ be b.l.f., then T is continuous.

Proof: Given $\epsilon > 0$, let $\delta = \epsilon/||T||$. Then, if $||[f_1] - [f_2]||_p < \delta$, then $|T([f_1]) - T([f_2])| = |T([f_1 - f_2])| \le ||T|| \cdot ||[f_1 - f_2]||_p < ||T|| \cdot \delta$ $= \epsilon$. So, in fact, T is uniformly continuous.

Theorem (Riesz Representation Theorem):
Let $1 \leq p < +\infty$ and let $(X, \mathfrak{A}, \mu)$ be σ-finite. If $T : L^p(X, \mathfrak{A}, \mu) \to \mathbb{R}$ is b.l.f., then there exists unique $[g] \in L^q(X, \mathfrak{A}, \mu)$ such that $T = T_g$ and $||g||_q = ||T||$.

Proof: Case $\mu(X)<+\infty$: Then, for all $E \in \mathfrak{A}$,
$\int_X |\mathcal{X}_E|^p d\mu = \mu(E) \leq \mu(X) < +\infty \Rightarrow \mathcal{X}_E \in \mathcal{L}^p$ for all $E \in \mathfrak{A}$.
Thus, $T([\mathcal{X}_E]) \in \mathbb{R}$ is defined.
Define $\nu : \mathfrak{A} \to \mathbb{R}$ via $\nu(E) = T([\mathcal{X}_E])$.
Claim: ν is a signed measure.
Proof of Claim:
First $|\nu(E)| = ||T(\mathcal{X}_E)|| \leq ||T||||\mathcal{X}_E||_p \leq ||T||\mu(X)^{1/p} < +\infty$.
Let $E = \bigcup_{n=1}^{\infty} E_n$, a disjoint union. We need $\nu(E) = \sum_{n=1}^{\infty} \nu(E_n)$.
Note that if E_1 and E_2 are disjoint, then $\mathcal{X}_{E_1 \cup E_2} = \mathcal{X}_{E_1} + \mathcal{X}_{E_2}$.
So, if $F_m = \bigcup_{n=1}^{m} E_n$, then $\mathcal{X}_{F_m} = \sum_{n=1}^{m} \mathcal{X}_{E_n}$. $||\mathcal{X}_E - \mathcal{X}_{F_m}||_p =$
$\left(\int_X |\mathcal{X}_E(x) - \mathcal{X}_{F_m}(x)|^p d\mu\right)^{1/p} = \left(\int_X |\mathcal{X}_{E \setminus F_m}(x)|^p d\mu\right)^{1/p} =$
$\mu(E \setminus F_m)^{1/p}$. We know, since $\mu(X) < +\infty \Rightarrow \mu(E) < +\infty \Rightarrow$
$\lim_m \mu(F_m) = \mu(E) \Rightarrow \lim_m \mu(E \setminus F_m) = \lim_n (\mu(E) - \mu(F_m))$.
So, $||\mathcal{X}_E - \mathcal{X}_{F_m}||_p \to 0$ as $n \to +\infty$. T is continuous by the last proposition. So, $\nu(E) = T([\mathcal{X}_E]) = \lim_m T([\mathcal{X}_{F_m}]) =$
$\lim_m \sum_{n=1}^{m} T([\mathcal{X}_{E_n}]) = \lim_m \sum_{n=1}^{m} \nu(E_n)$, and so $\nu(E) = \sum_{n=1}^{\infty} \nu(E_n)$, Thus, ν is a signed measure.

Recall Hahn-Decomposition: There exist $A \dot{\cup} B = X$, A-positive, B-negative, $\nu^+(E) = \nu(E \cap A)$, $\nu^-(E) = \nu(E \cap B)$, then ν^+ and ν^- are positive measures, and $\nu(E) = \nu^+(E) - \nu^-(E)$.
Let $\mu(E) = 0 \Rightarrow \mu(E \cap A) = 0 \Rightarrow [\mathcal{X}_{E \cap A}] = [0]$. Thus, $\nu^+(E) = \nu(E \cap A) = T([\mathcal{X}_{E \cap A}]) = 0$, and $\nu^+ \ll \mu$. Similarly $\nu^- \ll \mu$.
Therefore, by Radon-Nikodym, There exist g^+ and g^- such that $\nu^+(E) = \int_E g^+ d\mu$ and $\nu^-(E) = \int_E g^- d\mu$, and so $\nu(E) =$
$\int_E (g^+ - g^-) d\mu = \int_X \mathcal{X}_E (g^+ - g^-) d\mu$.

Let $g = g^+ - g^-$. Then, $T(\mathcal{X}_E) = \int_X \mathcal{X}_E g d\mu$, and so if $\psi = \sum_{k=1}^{n} \alpha_k \mathcal{X}_{E_k}$, then $T(\psi) = \sum_{k=1}^{n} \alpha_k T(\mathcal{X}_{E_k}) = \sum_{k=1}^{n} \int_X \mathcal{X}_{E_k} g d\mu = \int_X \psi g d\mu$. Now, we want to show $g \in \mathcal{L}^q$.

Take ϕ_n simple, $0 \leq \phi_1 \leq \phi_2 \leq \ldots \leq g^+$. Then, $\lim_n \phi_n(x) = g^+(x)$.

Similarly, take ψ_n simple, $0 \leq \psi_1 \leq \psi_2 \leq \ldots \leq g^-$.

Let $\gamma_n = \phi_n^{q/p} - \psi_n^{q/p}$. Then, γ_n is simple. Now, $T(\gamma_n) = \int_X \gamma_n g d\mu$ $= \int_X (\phi_n^{q/p} - \psi_n^{q/p})(g^+ - g^-) d\mu = \int_X (\phi_n^{q/p} g^+ + \psi_n^{q/p} g^-) d\mu$.

By the MCT, $\lim_n T(\gamma_n) = \int_X ((g^+)^{q/p}(g^+) + (g^-)^{q/p}(g^-)) d\mu = \int_X ((g^+)^{q/p+1} + (g^-)^{q/p+1}) d\mu = \int_X |g|^{q/p+1} d\mu = \int_X |g|^q d\mu$.

But, $||\gamma_n||_p = (\int_X |\gamma_n|^p d\mu)^{1/p} = (\int_X (\phi_n^q + \psi_n^q) d\mu)^{1/p} \xrightarrow[\text{By MCT}]{} (\int_X ((g^+)^q + (g^-)^q) d\mu)^{1/p} = (\int_X |g|^q d\mu)^{1/p}$.

Now, $|T(\gamma_n)| \leq ||T|| \cdot ||\gamma_n||_p$. So, $||g||_q^q = \int_X |g|^q d\mu \leq ||T|| (||g||_q)^{q/p} \Rightarrow ||g||_q^{q - q/p} \leq ||T|| \Rightarrow ||g||_q \leq ||T||$, in particular, $g \in \mathcal{L}^q$. Since $g \in \mathcal{L}^q \Rightarrow T_g$ is b.l.f. by the proposition.

$T_g(\psi) = \int_X \psi g d\mu = T(\psi)$, ψ simple, T_g and T b.l.f. $\Rightarrow T_g$ and T are continuous, by the previous proposition.

By Density Theorem, the simple functions are dense in L^p. So, two continuous functions are equal on dense set $\Rightarrow$ They are equal everywhere.

(Recall simple dense $\Rightarrow$ If $f \in \mathcal{L}^p$, then there exists $\{\psi_n\}$ such that $||f - \psi_n||_p \to 0$. If T and T_g are continuous, then $T(f) = \lim_n T(\psi_n)$ $= \lim_n T_g(\psi_n) = T_g(f)$.)

g is unique $a.e.$ follows from the earlier proposition.

<u>General Case X</u>: X is σ-finite $\Rightarrow X = \bigcup_{n=1}^{\infty} X_n$, X_n disjoint and $\mu(X_n) < +\infty$.

Now, if $f \in \mathcal{L}^p(X_n, \mathfrak{A}, \mu)$ and define $\widetilde{f} : X \to \mathbb{R}$ by

$$\widetilde{f}(x) = \begin{cases} f(x) & \text{if } x \in X_n \\ 0 & \text{if } x \notin X_n \end{cases}$$

then $\widetilde{f} \in \mathcal{L}^p(X, \mathfrak{A}, \mu)$ and $||\widetilde{f}||_p = ||f||_p$.

Given $f_1, f_2 \in \mathcal{L}^p(X_n, \mathfrak{A}, \mu)$, then $\widetilde{f_1} + \widetilde{f_2} = \widetilde{f_1 + f_2}$, and $\widetilde{\alpha f_1} = \alpha \widetilde{f_1}$. (Think of $\mathcal{L}^p(X_n, \mathfrak{A}, \mu)$ as a subspace of $\mathcal{L}^p(X, \mathfrak{A}, \mu)$)

Define $T_n : L^p(X_n, \mathfrak{A}, \mu) \to \mathbb{R}$ by $T_n([f]) = T([\widetilde{f}\ ])$.
Now, $|T_n([f])| = |T([\widetilde{f}\ ])| \le ||T|| \cdot ||\widetilde{f}\ ||_p = ||T|| \cdot ||f||_p$. So, T_n is b.l.f. on $L^p(X, \mathfrak{A}, \mu)$. Since $\mu(X_n) < +\infty$, by the first case, there exists $g_n \in \mathcal{L}^q(X_n, \mathfrak{A}, \mu)$ satisfies for all $f \in \mathcal{L}^p(X_n, \mathfrak{A}, \mu)$, $T_n(f) = \int_{X_n} f g_n d\mu$. Define $g : X \to \mathbb{R}^e$ by

$$g(x) = \begin{cases} g_1(x) & \text{if } x \in X_1 \\ g_2(x) & \text{if } x \in X_2 \\ \cdot & \cdot \\ \cdot & \cdot \end{cases}$$

If we take $f_j \in \mathcal{L}^p(X_j, \mathfrak{A}, \mu)$, and let $\widetilde{f_j} \in \mathcal{L}^p(X, \mathfrak{A}, \mu)$, then
$T\left(\sum_{j=1}^{n} \widetilde{f_j}\right) = \sum_{j=1}^{n} T(\widetilde{f_j}\ ) = \sum_{j=1}^{n} T_j(f_j) = \sum_{j=1}^{n} \int_{X_j} f_j g_j d\mu =$
$\int_X (\widetilde{f_1} + \widetilde{f_2} + \ldots + \widetilde{f_n}\ ) g d\mu$.
So, $T(f) = \int_X f g d\mu$ as long as f is nonzero only on finitely many X_j
Let

$$h_m(x) = \begin{cases} g_1(x) & \text{if } x \in X_1 \\ g_2(x) & \text{if } x \in X_2 \\ \cdot & \cdot \\ \cdot & \cdot \\ g_m(x) & \text{if } x \in X_m \\ 0 & \text{if } x \notin \bigcup_{j=1}^{m} X_j \end{cases}$$

Then, $|h_m(x)|^t \nearrow |g(x)|^t$, and if $F_m = \bigcup_{j=1}^{m} X_j$, then $h_m = g \cdot \mathcal{X}_{F_m}$.
Let $f_m(x) = sgn(g(x))|h_m(x)|^{q/p}$.
We need to show: (1) If $g(x) = g_n(x)$ for $x \in X$, then $[g] \in L^q(X, \mathfrak{A}, \mu)$. (2) For $[f] \in L^p(X, \mathfrak{A}, \mu)$, $T([f]) = \int_X f g d\mu$.
Now, $f_m = sgn(g)|g|^{q/p} \mathcal{X}_{F_m}$, which is nonzero only on $\bigcup_{n=1}^{m} X_n$.
So, $f_m = \sum_{n=1}^{m} sgn(g)|g|^{q/p} \mathcal{X}_{X_n}$, and $T(f_m) = \sum_{n=1}^{m} T(sgn(g)|g|^{q/p} \mathcal{X}_{X_n})$
$= \sum_{n=1}^{m} \int_X sgn(g)|g|^{q/p} \mathcal{X}_{X_n} d\mu = \sum_{n=1}^{m} \int_X |g|^{q/p+1} \mathcal{X}_{X_n} d\mu =$
$\int_X |g|^{q/p+1} \mathcal{X}_{F_m} d\mu \le ||T|| ||f_m||_p = ||T|| (\int_X |g|^q \mathcal{X}_{F_m} d\mu)^{1/p}$. So, $(\int_X |g|^q \mathcal{X}_{F_m} d\mu)^{1-1/p} \le ||T||$ for all m. Let $m \to +\infty$ and use MCT, then $(\int_X |g|^q d\mu)^{1/q} \le ||T|| \Rightarrow ||g||_q \le ||T||$.

Finally, we know that $T_g(f) = \int_X fgd\mu$ is a b.l.f. If f is only nonzero on, say X_n, then $T_g(f) = \int_{X_n} fgd\mu = \int_X fg_nd\mu = \int_X f(g\mathcal{X}_{X_n})d\mu = T(f)$. Hence, by taking sums, if f is only nonzero on $F_m = \bigcup_{n=1}^{m} X_n$, $T_g(f) = T(f)$.

Finally, given any $f \in \mathcal{L}^p$, by MCT, $f\mathcal{X}_{F_m} \underset{\|\cdot\|}{\rightarrow} f$. So,

$T_g(f) = \lim_m T_g(f\mathcal{X}_{F_m}) = \lim_m T(f\mathcal{X}_{F_m}) = T(f)$.