<u>HW 18</u>: ~~Let $g : [c,d] \to [a,b]$ be absolutely continuous and $F : [a,b] \to \mathbb{R}$ be absolutely continuous. Prove that $F \circ g : [c,d] \to \mathbb{R}$ is absolutely continuous.~~ DELETE

<u>HW 19</u> Let $f, g : [a,b] \to \mathbb{R}$ be absolutely continuous. Prove that: i) $f \cdot g$ is absolutely continuous

ii) $(f \cdot g)'(x) = f'(x)g(x) + f(x)g'(x)$ a.e.

iii) $\int_a^b f'g \, dm = f(b)g(b) - f(a)g(a) - \int_a^b fg' \, dm$

where $dm$ denotes Lebesgue measure

<u>HW 20</u>: ~~Set F(x) = Σ~~ Let $\{r_n\}$ be an enumeration of the rationals. Set $F(x) = \sum_{n=1}^{\infty} \frac{1}{2^n} \chi_{[r_n, +\infty)}(x)$

Show that: i) $x < y \Rightarrow F(x) < F(y)$

ii) $\lim_{x \to -\infty} F(x) = 0$, $\lim_{x \to +\infty} F(x) = 1$

iii) $F$ is continuous from the right

iv) If $x$ is irrational, then $F$ is continuous at $x$

**Theorem (Riesz-Fischer):**
For $1 \leq p \leq +\infty$, $L^p(X, \mathfrak{A}, \mu)$ is a complete metric space, i.e., a Banach space

Proof: When $p \neq +\infty$. Let $\{[f_n]\}_{n=1}^{\infty}$ be Cauchy sequence in $L^p$.
(We need to show: There exists $[f] \in L^p$ such that $||[f] - [f_n]||_p = ||f - f_n||_p \to 0$ as $n \to +\infty$.) Inductively define $N_1 \leq N_2 \leq \ldots$ so that for $n, m \geq N_k$, $||[f_n] - [f_m]|| \leq 1/2^k$. Define $g_1 = f_{N_1}$, $g_2 = f_{N_2} - f_{N_1}, \ldots, g_k = f_{N_k} - f_{N_{k-1}}$. Then, $||g_k||_p \leq 1/2^{k-1}$, $k > 1$
$\Rightarrow \sum_{k=1}^{\infty} ||g_k||_p < +\infty \Rightarrow \sum_{k=1}^{\infty} ||[g_k]||_p < +\infty$. Let $\sum_{k=1}^{\infty} ||[g_k]||_p = C$.

Let $h_m(x) = \sum_{k=1}^{m} |g_k(x)|$. Then, $h_1(x) \leq h_2(x) \leq \ldots \leq h(x)$ where
$h(x) = \sum_{k=1}^{\infty} |g_k(x)|$, $h(x) = \lim_m h_m(x)$.

Now, $||h_m(x)||_p = ||\sum_{k=1}^{m} |g_k(x)|||_p \leq \sum_{k=1}^{m} |||g_k(x)|||_p \leq C$.
So, $\int_X |h_m(x)|^p d\mu \leq C^p < +\infty$.
Now, $|h_m(x)|^p = h_m(x)^p \nearrow h(x)^p$. So by MCT, $\int_X |h(x)|^p d\mu = \lim_m \int_X |h_m(x)|^p d\mu \leq C^p \Rightarrow \mu(\{x : h(x) = +\infty\}) = 0$.
So, let $Y = \{x : h(x) \neq +\infty\}$. Then, $\mu(X \backslash Y) = 0$. So, for $x \in Y$, $\sum_{k=1}^{\infty} |g_k(x)| < +\infty$, and so for $x \in Y$, $\sum_{k=1}^{\infty} g_k(x)$ converges.

Define $f(x) = \begin{cases} \sum_{k=1}^{\infty} g_k(x) & \text{if } x \in Y \\ 0 & \text{if } x \notin Y \end{cases}$

Now, $||f(x) - \sum_{k=1}^{m} g_k(x)||_p = \left( \int_X |f(x) - \sum_{k=1}^{m} g_k(x)|^p d\mu \right)^{1/p} =$

$\left( \int_Y |f(x) - \sum_{k=1}^{m} g_k(x)|^p d\mu \right)^{1/p} = \left( \int_Y |\sum_{k=m+1}^{\infty} g_k(x)|^p d\mu \right)^{1/p} =$

$||\sum_{k=m+1}^{\infty} g_k(x)||_p \leq \sum_{k=m+1}^{\infty} ||g_k(x)||_p \to 0$ as $m \to +\infty$.

$\sum_{k=1}^{m} g_k = f_{N_1} + (f_{N_2} - f_{N_1}) + \ldots + (f_{N_m} - f_{N_{m-1}}) = f_{N_m}$. So,
$||f - f_{N_m}||_p \to 0$ as $m \to +\infty$, and $||[f] - [f_{N_m}]||_p \to 0$ as $m \to +\infty$.

Claim: $||[f]-[f_n]||_p \to$ as $m \to +\infty$.
Proof of Claim: Given $\epsilon > 0$, pick $m_1$ such that $m > m_1 \Rightarrow$ $||[f]-[f_{N_m}]||_p < \epsilon/2$, and pick $m_2$ such that $1/2^{m_2} < \epsilon/2$.
Let $N = max\{N_{m_1}, N_{m_2}\}$. Let $m \geq m_1, m_2; n, N_m \geq N_{m_2}$. Then, for $n \geq N$, $||[f] - [f_n]||_p \leq ||f - f_{N_m}||_p + ||f_n - f_{N_m}||_p <$ $\epsilon/2 + 1/2^{m_2} < \epsilon$.

**Special Examples of $L^p$**

(1) Let $X = [0,1], \mathfrak{A} = \mathfrak{M}, \mu = \lambda$
We usually write $L^p([0,1], \mathfrak{M}, \lambda) \equiv L^p[0,1]$.

(2) Let $X = \{1, 2, 3, \ldots, n\}, \mathfrak{A} =$ all subsets, $\mu =$ counting measure
$f = g\, a.e.\mu \Leftrightarrow f(x) = g(x)$ for all $x$
$f \longleftrightarrow (f(1), f(2), \ldots, f(n)) \in \mathbb{R}^n$

$$(\int_X |f(x)|^p d\mu)^{1/p} = \left(\sum_{j=1}^{n} |f(j)|^p\right)^{1/p}$$

$f \longleftrightarrow (\alpha_1, \alpha_2, \ldots, \alpha_n)$

$$||f||_p = ||(\alpha_1, \alpha_2, \ldots, \alpha_n)||_p = \left(\sum_{j=1}^{n} |\alpha_j|^p\right)^{1/p}$$

$L^p(\{1, 2, \ldots, n\}, \mathfrak{A}, \mu) \longleftrightarrow l_n^p$

$$l_n^p = \{(\alpha_1, \alpha_2, \ldots, \alpha_n) \in \mathbb{R}^n : ||(\alpha_1, \alpha_2, \ldots, \alpha_n)||_p = \left(\sum_{j=1}^{n} |\alpha_j|^p\right)^{1/p} \}$$

So, if we take $\mathbb{R}^n$, define $||\cdot||_p$ as above, then that is a norm on $\mathbb{R}^n$.

That is $$\left(\sum_{j=1}^{n} |\alpha_j + \beta j|^p\right)^{1/p} \leq \left(\sum_{j=1}^{n} |\alpha_j|^p\right)^{1/p} + \left(\sum_{j=1}^{n} |\beta_j|^p\right)^{1/p}$$

In particular, when $p = 2$, note that this is the usual **Euclidean norm**

$$||(\alpha_1, \alpha_2, \ldots, \alpha_n)|| = \sqrt{\sum_{j=1}^{n} |\alpha_j|^2}$$

Consider $f \in L^p(\{1, 2, \ldots, n\}, \mathfrak{A}, \mu), g \in L^q(\{1, 2, \ldots, n\}, \mathfrak{A}, \mu)$

$$\int_X f g d\mu = \sum_{j=1}^{n} f(j) g(j)$$

$f \longleftrightarrow (\alpha_1, \alpha_2, \ldots, \alpha_n), g \longleftrightarrow (\beta_1, \beta_2, \ldots, \beta_n)$

$$\left|\sum_{j=1}^{n} \alpha_j \beta_j\right| \leq \left(\sum_{j=1}^{n} |\alpha_j|^p\right)^{1/p} + \left(\sum_{j=1}^{n} |\beta_j|^q\right)^{1/q} \quad 1/p + 1/q = 1$$

(3) Let $X = \mathbb{N} = \{1, 2, 3, \ldots\}$, $\mathfrak{A} =$ all subsets, $\mu =$ counting measure
$f = g\, a.e.\mu \Leftrightarrow f(n) = g(n)$ for all $n$
$f \longleftrightarrow (f(1), f(2), f(3), \ldots)$ $\infty$-tuple
$f \in L^p(\mathbb{N}, \mathfrak{A}, \mu) \Leftrightarrow \int_{\mathbb{N}} |f(x)|^p d\mu = \sum_{n=1}^{\infty} |f(n)|^p < +\infty$

$L^p(\mathbb{N}, \mathfrak{A}, \mu) \longleftrightarrow l^p = \{(\alpha_1, \alpha_2, \alpha_3, \ldots) : \sum_{j=1}^{\infty} |\alpha_j|^p < +\infty\}$

Minkowski: If $\sum_{n=1}^{\infty} |\alpha_n|^p < +\infty$ and $\sum_{n=1}^{\infty} |\beta_n|^p < +\infty$, then
$\sum_{n=1}^{\infty} |\alpha_n + \beta_n|^p < +\infty$ and $\left(\sum_{n=1}^{\infty} |\alpha_n + \beta_n|^p\right)^{1/p} \leq$
$\left(\sum_{n=1}^{\infty} |\alpha_n|^p\right)^{1/p} + \left(\sum_{n=1}^{\infty} |\beta_n|^p\right)^{1/p}$

Holder: If $\sum_{n=1}^{\infty} |\alpha_n|^p < +\infty$ and $\sum_{n=1}^{\infty} |\beta_n|^p < +\infty$, then
$\left|\sum_{n=1}^{\infty} \alpha_n \beta_n\right| \leq \left(\sum_{n=1}^{\infty} |\alpha_n|^p\right)^{1/p} + \left(\sum_{n=1}^{\infty} |\beta_n|^q\right)^{1/q}$

Riesz-Fisher: $l^p$ together with $||(\alpha_1, \alpha_2, \alpha_3, \ldots)||_p = \left(\sum_{n=1}^{\infty} |\alpha_n|^p\right)^{1/p}$
is a **norm** and the **metric** that it induces is **complete**.
When $p = +\infty$, $L^\infty(\mathbb{N}, \mathfrak{A}, \mu)$, $||f||_\infty = sup\{f(n) : n = 1, 2, 3, \ldots\}$
So, $L^\infty(\mathbb{N}, \mathfrak{A}, \mu) \longleftrightarrow l^\infty = \{(\alpha_1, \alpha_2, \alpha_3, \ldots) : sup|\alpha_n| < +\infty\}$,
$||(\alpha_1, \alpha_2, \alpha_3, \ldots)||_\infty = \sup_n |\alpha_n|$