1. The L^p-spaces

1.1 Minkowski-Holder inequality

Definition: Let $(X, \mathfrak{A}, \mu)$ be a measure space. We let $\mathcal{L}^p(X, \mathfrak{A}, \mu)$, for $1 \leq p < +\infty$, denote the set of measurable functions on X such that $\int_X |f|^p d\mu < +\infty$. For $f \in \mathcal{L}^p(X, \mathfrak{A}, \mu)$, we set $||f||_p = (\int_X |f|^p d\mu)^{1/p}$. Then, $||\cdot||_p$ is called the ***p* norm of *f***.

Definition: Let $f : X \to \mathbb{R}^e$ be measurable, we say that f is **essentially bounded** if there exists $M > 0$ such that $\mu(\{x : |f(x)| > M\}) = 0$. The least such M is called the **essential supremum** of f, denoted by $||f||_\infty = \underset{x\in X}{ess\ sup}\ |f|$.

Examples:

(1) Let $X = \mathbb{R}$, $f : X \to \mathbb{R}^e$, $f(x) = \begin{cases} 1/x & \text{if } x \neq 0 \\ +\infty & \text{if } x = 0 \end{cases}$
Notice that $\lambda(\{x : |f(x)| = +\infty\}) = 0$. But, f is <u>not</u> essentially bounded because given any $M > 0$, $\lambda(\{x : |f(x)| > M\}) = \lambda((-1/M, 1/M)) = 2/M > 0$.

(2) Let $X = \mathbb{R}$, $f(x) = \begin{cases} 1 & \text{if } x \text{ is irrational} \\ x & \text{if } x \text{ is rational} \end{cases}$
Then, f is not bounded, but f is essentially bounded and $ess\ sup|f| = +1$.

Definition: $\mathcal{L}^\infty(X, \mathfrak{A}, \mu)$ is the **set of essentially bounded measurable functions.**

Definition: Let $1 \leq p \leq +\infty$ and $1 \leq q \leq +\infty$. Then, p and q are called the **Holder conjugates** if $1/p + 1/q = 1$.

Special cases: $p = 1, q = +\infty$ or $p = +\infty, q = 1$
$p = 2, q = 2$
$p = 3, q = 3/2$
.
$p = p, q = p/(p-1)$
$1/p + 1/q = 1 \Rightarrow q + p = pq \Rightarrow q - pq = -p \Rightarrow (1-p)q = -p \Rightarrow q = -p/(1-p) \Rightarrow q = p/(p-1)$

Lemma: If $1/p + 1/q = 1$, $a \geq 0$ and $b \geq 0$, then $a \cdot b \leq a^p/p + b^q/q$.

Proof: Let $h(t) = t^p/p + b^q/q - tb$ where $t \geq 0$ (show that h is nonnegative and so the minimum value is nonnegative.)
$h'(t) = t^{p-1} - b = 0 \Leftrightarrow t^{p-1} = b \Leftrightarrow t = b^{1/(p-1)}$ is the critical value.
$h''(t) = (p-1)t^{p-2} \geq 0$ since $p > 1$ and $t \geq 0$. So, the absolute minimum occurs at $t = b^{1/(p-1)}$ and $h(b^{1/(p-1)}) = ((b^{1/(p-1)})^p)/p + b^q/q - b^{1/(p-1)}b = b^q(1/p + 1/q) - b^q = b^q(1/p + 1/q - 1) = b^p \cdot 0 = 0$.

Theorem (Holders Inequality):
Let $1 \leq p, q \leq +\infty$ and $1/p + 1/q = 1$. If $f \in \mathcal{L}^p(X, \mathfrak{A}, \mu)$ and $g \in \mathcal{L}^q(X, \mathfrak{A}, \mu)$, then $f \cdot g \in \mathcal{L}^1(X, \mathfrak{A}, \mu)$ and $||fg||_1 \leq ||f||_p \cdot ||g||_q$.

Proof: Case $p = 1$ and $q = +\infty$

Now, $g \in \mathcal{L}^\infty$. Let $C = ||g||_\infty$.
Then, $\mu(\{x : |g(x)| > C\}) = 0$.
Let $N = \{x : |g(x)| > C\}$, so for $x \in X \backslash N$, $|g(x)| \leq C$.
So, $||fg||_1 = \int_X |f(x)g(x)| d\mu = \int_{X\backslash N} |f(x)g(x)| d\mu \leq \int_{X\backslash N} C|f(x)| d\mu = C\int_{X\backslash N} |f(x)| d\mu = C\int_X |f(x)| d\mu = C \cdot ||f||_1 = ||g||_\infty \cdot ||f||_1$.

Similar for the case $p = +\infty$ and $q = 1$.

Case $1 < p, q < +\infty$

Set $a = |f(x)|/||f||_p$ and $b = |g(x)|/||g||_q$.
Then, by Lemma, $(|f(x)||g(x)|)/(||f||_p ||g||_q) \leq (|f(x)|^p/||f||_p^p)/p + (|g(x)|^q/||g||_q^q)/q$.
So, $\int_X (|f(x)||g(x)|)/(||f||_p ||g||_q) d\mu \leq \int_X [(|f(x)|^p/||f||_p^p)/p + (|g(x)|^q/||g||_q^q)/q] d\mu$
$\Rightarrow 1/(||f||_p ||g||_q) \int_X |f(x)g(x)| d\mu \leq 1/p||f||_p^p \int_X |f(x)|^p d\mu + 1/q||g||_q^q \int_X |g(x)|^q d\mu$
$\Rightarrow ||fg||/(||f||_p ||g||_q \leq 1/p + 1/q = 1$
$\Rightarrow ||fg|| \leq ||f||_p ||g||_q$

Proposition: For $1 \leq p \leq +\infty$, $\mathcal{L}^p(X, \mathfrak{A}, \mu)$ is a real vector space. (i.e. $f, g \in \mathcal{L}^p \Rightarrow f + g \in \mathcal{L}^p$; $\alpha f \in \mathcal{L}^p$ for any $\alpha \in \mathbb{R}$.)

Proof: Let $f \in \mathcal{L}^p$ and $\alpha \in \mathbb{R}$ (show $\alpha f \in \mathcal{L}^p$.)

Suppose that $p \neq +\infty$.Then, $\int_X |f|^p d\mu < +\infty$. Hence, $\int_X |\alpha f|^p d\mu = |\alpha|^p \int_X |f|^p d\mu < +\infty$, and so $\alpha f \in \mathcal{L}^p$.

$\Rightarrow ||\alpha f||_p = |\alpha| \cdot ||f||_p$

Suppose that $p = +\infty$, then $f \in \mathcal{L}^\infty$. Then, there exists M such that $\mu(\{x : |f(x)| > M\}) = 0$

$\Rightarrow \mu(\{x : |\alpha f(x)| > |\alpha| M\}) = 0$. So, αf is essentially bounded by $|\alpha| M$.

$\Rightarrow ||\alpha f||_\infty = |\alpha| ||f||_\infty$

Let $f, g \in \mathcal{L}^p$ (now, show that $f + g \in \mathcal{L}^p$.)

If $p = +\infty$, then there exist M_1 and M_2 such that $\mu(\{x : |f(x)| > M_1\}) = 0$ and $\mu(\{x : |g(x)| > M_2\}) = 0 \Rightarrow$ $\mu(\{x : |f(x) + g(x)| > M_1 + M_2\}) = 0$.

So, $f + g$ is essentially bounded by $M_1 + M_2$. Thus, $f, g \in \mathcal{L}^\infty \Rightarrow f + g \in \mathcal{L}^\infty$.

$\Rightarrow ||f + g||_\infty \leq ||f||_\infty + ||g||_\infty$

If $p < +\infty$, note that

$$|f(x) + g(x)| \leq \begin{cases} |f(x)| + |f(x)| & \text{when } |f(x)| \geq |g(x)| \\ |g(x)| + |g(x)| & \text{when } |g(x)| \geq |f(x)| \end{cases}$$

So, $|f(x) + g(x)|^p \leq \begin{cases} 2^p |f(x)|^p & \text{when } |f(x)| \geq |g(x)| \\ 2^p |g(x)|^p & \text{when } |g(x)| \geq |f(x)| \end{cases}$

Thus, $|f(x) + g(x)|^p \leq 2^p |f(x)|^p + 2^p |g(x)|^p$.

So, if $f, g \in \mathcal{L}^p$, then $\int_X |f(x) + g(x)| d\mu \leq$ $\int_X 2^p (|f(x)|^p + |g(x)|^p) d\mu < +\infty$. Thus, $f + g \in \mathcal{L}^p$.

$\Rightarrow ||f + g||_p^p \leq 2^p [||f||_p^p + ||g||_p^p]$

$\Rightarrow ||f + g||_p \leq 2[||f||_p^p + ||g||_p^p]^{1/p}$

Theorem (Minkowski's Inequality):
Let $1 \leq p \leq +\infty$. If $f, g \in \mathcal{L}^p(X, \mathfrak{A}, \mu)$, then $f + g \in \mathcal{L}^p(X, \mathfrak{A}, \mu)$ and $||f+g||_p \leq ||f||_p + ||g||_p$.

Proof: Case when $p = +\infty$ was done previously in Proposition.
When $p = 1$, $||f+g||_1 = \int_X |f(x) + g(x)| d\mu \leq$
$\int_X (|f(x)| + |g(x)|) d\mu = ||f||_1 + ||g||_1$
When $1 < p < +\infty$, $||f+g||_p^p = \int_X |f(x) + g(x)|^p d\mu \leq$
$\int_X |f(x) + g(x)|^{p-1} (|f(x)| + |g(x)|) d\mu \leq$
$\int_X |f(x)||f(x) + g(x)|^{p-1} d\mu + \int_X |g(x)||f(x) + g(x)|^{p-1} d\mu$
$\leq \left(\int_X |f(x)|^p d\mu\right)^{1/p} \left(\int_X (|f(x) + g(x)|^{p-1})^q d\mu\right)^{1/q} +$
$\left(\int_X |g(x)|^p d\mu\right)^{1/p} \left(\int_X (|f(x) + g(x)|^{p-1})^q d\mu\right)^{1/q}$
Recall that $q + p = pq \Rightarrow p = pq - q \Rightarrow p = (p-1)q$.
So, $||f+g||_p^p \leq (||f||_p + ||g||_p) \left(\int_X |f(x) + g(x)|^p d\mu\right)^{1/q} \Rightarrow$
$\left(\int_X |f(x) + g(x)|^p d\mu\right)^1 \leq$
$(||f||_p + ||g||_p) \left(\int_X |f(x) + g(x)|^p d\mu\right)^{1/q} \Rightarrow$
$\left(\int_X |f(x) + g(x)|^p d\mu\right)^{1-1/q} \leq ||f||_p + ||g||_p \Rightarrow$
$\left(\int_X |f(x) + g(x)|^p d\mu\right)^{1/p} \leq ||f||_p + ||g||_p \Rightarrow$
$||f+g||_p \leq ||f||_p + ||g||_p$

Note: Let $f \in \mathcal{L}^p(X, \mathfrak{A}, \mu)$. Then, $||f||_p = 0 \Leftrightarrow \int_X |f(x)|^p d\mu = 0 \Leftrightarrow$ $|f(x)|^p = 0\, a.e.\mu \Leftrightarrow f = 0\, a.e.\mu$. So, $||f||_p = 0 \Leftrightarrow f = 0\, a.e.\mu$.

1.2 Convergence and completeness (Riesz-Fischer)

Definition: Let V be a vector space over $\mathbb{R}$. Then a function $||\cdot|| : V \to \mathbb{R}$ is called a **norm on V** provided:

(i) $||v|| \geq 0$ for all $v \in V$
(ii) $||v|| = 0 \Leftrightarrow v$ for all $v \in V$
(iii) $||\alpha v|| = |\alpha|||v||$ for any $\alpha \in \mathbb{R}$ and $v \in V$
(iv) $||v + w|| \leq ||v|| + ||w||$ *Triangle Inequality*

Example: Let $V = \mathbb{R}^n = \{v = (x_1, x_2, \ldots, x_n) : x_i \in \mathbb{R}\}$. Then **Euclidean norm**, $||v|| = \left(\sum_{i=1}^{n} |x_i|^2\right)^{1/2}$ satisfies above properties.

Note: A norm is good for defining a distance or a metric on a vector space.

Proposition: If $||\cdot||$ is a norm on V, then setting $\rho(v, w) = ||v - w||$ defines a metric on V, called the metric induced by the norm

Proof: (i) $\rho(v, w) \geq 0$ is true since $||v - w|| \geq 0$.
(ii) $\rho(v, w) = 0 \Leftrightarrow v = w$ by (ii) of the definition of norm.
(iii) $\rho(v, w) = ||v - w|| = ||(-1)(w - v)|| = |-1|||w - v|| = ||w - v|| = \rho(w, v)$.
(iv) $\rho(v, w) = ||v - w|| = ||v - z + z - w|| \leq ||v - z|| + ||z - w|| = \rho(v, z) + \rho(z, w)$

Note: Metrics that are defined by norms are translation invariant (preserve distance.)

Propn: $\rho : V \times V \to \mathbb{R}$ metric, that is translation invariant $\rho(x+z, y+z) = \rho(x,y)$, scale invariant $\rho(\lambda x, \lambda y) = |\lambda| \rho(x,y)$, then $||x|| = \rho(0,x)$ is a norm.

Pf: ~~[illegible]~~

Defn A normed space $(V, ||\cdot||)$ is called a Banach space, if (V, ρ) is complete

Note: p-norm satisfies (i), (iii) and (iv) of the definition of norm. So, make p-norm into a norm, we need to introduce an equivalence relation:

(1) Given $f, g \in \mathcal{L}^p(X, \mathfrak{A}, \mu)$, we write $f \sim g \Leftrightarrow f - g = 0 \, a.e. \mu$.
Then: $f \sim f$ since $f - f = 0$ *reflexive*
$f \sim g \Leftrightarrow f - g = 0 \, a.e. \mu \Leftrightarrow g - f = 0 \, a.e. \mu \Leftrightarrow g \sim f$ *symmetric*
$f \sim g$ and $g \sim h \Rightarrow f - g = 0 \, a.e. \mu$ and $g - h = 0 \, a.e. \mu \Rightarrow$
$f - h = (f - g) + (g - h) = 0 \, a.e. \mu \Rightarrow f \sim h$ *transitive*
So, $\sim$ is an equivalence relation.

(2) Next, let $f \sim g$ and let $N = \{x : f(x) - g(x) \neq 0\}$. Then, $\mu(N) = 0$.
If $1 \leq p < +\infty$, then $||f||_p^p = \int_X |f(x)|^p d\mu = \int_{X \setminus N} |f(x)|^p d\mu + \int_N |f(x)|^p d\mu = \int_{X \setminus N} |f(x)|^p d\mu = \int_{X \setminus N} |g(x)|^p d\mu = ||g||_p^p$.
So, $f \sim g \Rightarrow ||f||_p = ||g||_p$.
Similarly, $f \sim g \Rightarrow ||f||_\infty = ||g||_\infty$.

(3) Finally, let $f_1 \sim g_1$, $f_2 \sim g_2$, $N_1 = \{x : f_1(x) \neq g_1(x)\}$, and $N_2 = \{x : f_2(x) \neq g_2(x)$. Then, $\mu(N_1) = \mu(N_2) = 0$. So, $\{x : f_1(x) + f_2(x) \neq g_1(x) + g_2(x)\} \subseteq N_1 \cup N_2 \Rightarrow f_1 + f_2 \sim g_1 + g_2$.

Definition: $L^p(X, \mathfrak{A}, \mu) = \{[f] : f \in \mathcal{L}^p(X, \mathfrak{A}, \mu)\}$ where $[f] = \{g : f \sim g\}$, equivalence class of f.

Note: If we set $[f_1] + [f_2] = [f_1 + f_2]$ by (3) above, this is a well-defined addition. Similarly, if $\alpha \in \mathbb{R}$ and $f \sim g \Rightarrow \alpha f \sim \alpha g$. Set $\alpha[f] = [\alpha f]$. These operations make $L^p(X, \mathfrak{A}, \mu)$ a **vector space**.

Note: Also, setting $||[f]||_p = ||f||_p$. This is well-defined by (2).

(i) $||[f]||_p = ||f||_p \geq 0$
(ii) $||[f]||_p = 0 \Leftrightarrow ||f||_p = 0 \Leftrightarrow f = 0 \, a.e. \mu \Leftrightarrow [f] = [0]$
(iii) $|\alpha| ||[f]||_p = |\alpha| ||f||_p = ||\alpha f||_p = ||[\alpha f]||_p = ||\alpha [f]||_p$
(iv) $||[f] + [g]||_p = ||[f + g]||_p = ||f + g||_p \leq ||f||_p + ||g||_p = ||[f]||_p + ||[g]||_p$

Thus, $(L^p(X, \mathfrak{A}, \mu), || \cdot ||_p)$ is a **normed space**.