Thus, F is bounded $\Rightarrow T_F \pm F$ are bounded $\Rightarrow H_1$ and H_2 are both bounded.

Proof of (c): We know that all of these limits exist for bounded and increasing functions. Thus, they exist for H_1 and H_2, and so they exist for $F = H_1 \quad H_2$.

Proof of (d): We know that H_1 and H_2 are continuous except at a countable set by Theorem 3.23 $\Rightarrow F = H_1 - H_2$ is continuous except for at most countably many points.

Proof of (e): Argue as (d) using Theorem 3.23.

Recall that if $F \in BV$ and T_F is a total variation, then $F = (T_F + F)/2 - (T_F - F)/2$. This decomposition of F is called the **Jordan decomposition of F**, and $(T_F + F)/2$ is called the **positive variation of F** and $(T_F - F)/2$ is called the **negative variation of F**.

Definition: A function $F : \mathbb{R} \to \mathbb{C}$ is **absolutely continuous** if for all $\epsilon > 0$, there exists $\delta > 0$ such that for all n and for all disjoint intervals (a_1, b_1), $(a_2, b_2), \cdots, (a_n, b_n)$ whenever $\sum_{i=1}^{n}(b_i - a_i) < \delta$, $\sum_{i=1}^{n}|F(b_i) - F(a_i)| < \epsilon$.
We say that $F : [a, b] \to \mathbb{C}$ is **absolutely continuous** if the above holds for all $(a_i, b_i) \subseteq [a, b]$. Note: absolute continuity $\Rightarrow$ uniform continuity

2) F, G AC $\Rightarrow$ $F+G$ AC, λF AC

Proposition 3.32: Let $H : \mathbb{R} \to \mathbb{R}$ be increasing, bounded and right continuous. Then, $\mu_H \ll m$ if and only if H is absolutely continuous.

Proof ($\Rightarrow$): Let $\mu_H \ll m$. Then, by Theorem 3.8 (The Lebesgue-Radon-Nikodym Theorem), there exists $f \geq 0$ such that $H(b) - H(a) = \int_{(a,b)} f dm$ for all a and b. Now, since H is bounded, $\lim_{x\to+\infty} H(x) = H(+\infty) < +\infty$ and $\lim_{x\to-\infty} H(x) = H(-\infty) < -\infty \Rightarrow \int_{\mathbb{R}} f dm = H(+\infty) - H(-\infty) < +\infty \Rightarrow f \in L^1(m)$. By Theorem 3.5 or Corollary 3.6, if $f \in L^1(m)$ and given $\epsilon > 0$, there exists $\delta > 0$ such that $m(E) < \delta \Rightarrow |\int_E f dm| < \epsilon$. Now, given $(a_1, b_1), (a_2, b_2), \cdots, (a_n, b_n)$ disjoint and $\sum_{i=1}^{n}(b_i - a_i) < \delta$. Let $E = \bigcup_{i=1}^{n}(a_i, b_i)$, then

$m(E) < \delta \Rightarrow |\int_E f dm| < \epsilon$. But, $\int_E f dm = \sum_{i=1}^{n}\int_{(a_i,b_i)} f dm =$

$\sum_{i=1}^{n} H(b_i) - H(a_i) = \sum_{i=1}^{n}|H(b_i) - H(a_i)| < \epsilon$. Thus, H is absolutely continuous.

Proof ($\Leftarrow$): Conversely, suppose that H is absolutely continuous, then given $\epsilon > 0$, let $\delta > 0$ such that for all n and for all disjoint intervals (a_1, b_1), (a_2, b_2), $\cdots$, (a_n, b_n) and $\sum_{i=1}^{n}(b_i - a_i) < \delta$ $\Rightarrow \sum_{i=1}^{n} H(b_i) - H(a_i) < \epsilon \Rightarrow \sum_{i=1}^{n} \mu_H((a_i, b_i]) = \mu_H(\bigcup_{i=1}^{n}(a_i, b_i])$ $< \epsilon$. Thus, $m(\bigcup_{i=1}^{n}(a_i, b_i]) < \delta \Rightarrow \mu_H(\bigcup_{i=1}^{n}(a_i, b_i]) < \epsilon$, and so $\mu_H \ll m$ by Theorem 3.5.

<u>Note</u>: Let $F : \mathbb{R} \to \mathbb{R}$ defined by $F(x) = x$. Then, F is absolutely continuous (to see this, let $\epsilon = \delta$), but $F \notin BV$.

Lemma 3.34: If F is absolutely continuous on $[a, b]$, then $F \in BV([a, b])$.

Proof: Let F be absolutely continuous on $[a, b]$. Let $\epsilon = 1$. Then, there exists $\delta > 0$ such that $\sum_{i=1}^{n}(b_i - a_i) < \delta \Rightarrow$ $\sum_{i=1}^{n}|F(b_i) - F(a_i)| < 1$. Now, find M such that $a + M\delta \leq b$ $< a + (M+1)\delta$. Given any partition, $a = x_0 < x_1 < \cdots$ $< x_n = b$, $\sum_{i=1}^{n}|F(x_i) - F(x_{i-1})|$ goes up if we add more points, so include the points $a + k\delta$ where $M \leq k < M + 1$; $a = x_0 < x_1 < x_2 < a + \delta < x_3 < \cdots < a + k\delta < x_n = b$. The sum of the length of these intervals must be most δ, and also for an example $|F(x_1) - F(x_0)| + |F(x_2) - F(x_1)| +$ $|F(a+\delta) - F(x_2)| < 1$. Thus, $\sum_{i=1}^{n}|F(x_i) - F(x_{i-1})| \leq$ $M + 1 = [(b-a)/\delta] + 1$ where $[(b-a)/\delta]$ is the greatest integer of $(b-a)/\delta$.

Lemma: Let F be absolutely continuous on $[a, b]$, then $T_F(x)$ is also absolutely continuous.

Proof: Let F be absolutely continuous on $[a, b]$. Given $\epsilon > 0$, pick the $\delta > 0$ that works for F and $\epsilon/2$. Suppose that (a_1, b_1), (a_2, b_2), $\cdots$, (a_n, b_n) are disjoint and $\sum_{i=1}^{n}(b_i - a_i) < \delta$. Without loss of generality, consider $a \leq a_1 < b_1 \leq a_2 < b_2 \leq \cdots < b_n \leq b$ and take $a_i = x_0^i < x_1^i < \cdots < x_{m_i}^i = b_i$ such that

$$|T_F(b_i) - T_F(a_i)| \le \sum_{j=1}^{m_i} |F(x^i_j) - F(x^i_{j-1})| + \epsilon/2n \Rightarrow$$

$$\sum_{i=1}^{n} |T_F(b_i) - T_F(a_i)| \le \sum_{i=1}^{n} \left(\sum_{j=1}^{m_i} |F(x^i_j) - F(x^i_{j-1})| + \epsilon/2n\right) =$$

$$\left(\sum_{i=1}^{n}\sum_{j=1}^{m_i} |F(x^i_j) - F(x^i_{j-1})|\right) + \epsilon/2 < \epsilon/2 + \epsilon/2 = \epsilon.$$ Thus, T_F is absolutely continuous.

INSERT 3.22-3.1, 3.2

Theorem 3.35 (The Fundamental Theorem of Calculus for Lebesgue Integrals): If $-\infty < a < b < +\infty$, and $F : [a,b] \to \mathbb{C}$, then the following are equivalent:

(a) F is absolutely continuous on $[a,b]$.

(b) There exists $f \in L^1([a,b])$ such that $F(x) - F(a) = \int_{[a,x]} f\,dm$, $\forall\, a \le x \le b$

(c) F' exists *a.e.*, $F' \in L^1$ and $F(x) - F(a) = \int_{[a,x]} F'\,dm$, $\forall\, a \le x \le b$

Proof of (a) $\Rightarrow$ (b): Let F be absolutely continuous on $[a,b]$, then $F \in BV([a,b])$ by Lemma 3.34. So, write $F = (T_F + F)/2 - (T_F - F)/2 = H_1 - H_2$. Then, H_1 and H_2 are increasing on $[a,b]$ and absolutely continuous by the previous Lemma. Now, extend these H_is to $\mathbb{R}$ by setting

$$\widetilde{H}_i(x) = \begin{cases} H_i(a) & \text{if } x \le a \\ H_i(x) & \text{if } a \le x \le b \\ H_i(b) & \text{if } x \ge b \end{cases}$$

Then, $\widetilde{H}_i$ is increasing, absolutely continuous and bounded $\Rightarrow$ $\mu_{\widetilde{H}_i} \ll m$. So, there exist $f_1, f_2 \in L^1(m)$ such that $\mu_{\widetilde{H}_i}(E) = \int_E f_i\,dm$. Let $f = f_1 - f_2$. Then, $\int_{(a,x)} f\,dm = \int_{(a,x)} f_1\,dm - \int_{(a,x)} f_2\,dm = (\widetilde{H}_1(x) - \widetilde{H}_1(a)) - (\widetilde{H}_2(x) - \widetilde{H}_2(a)) = (\widetilde{H}_1(x) - \widetilde{H}_2(x)) - (\widetilde{H}_1(a) - \widetilde{H}_2(a)) = (\widetilde{H}_1 - \widetilde{H}_2)(x) - (\widetilde{H}_1 - \widetilde{H}_2)(a) = F(x) - F(a)$.

Proof of (b) $\Rightarrow$ (c): ~~Let $f \in L^1$ such that $F(x) - F(a) = \int_{(a,x)} f\,dm$. If $y > x$, $(F(y) - F(x))/(y - x) = 1/(y-x)\int_{(x,y)} f\,dm \to f(x)$ *a.e.* Thus, $\lim_{y\to x}(F(y) - F(x))/(y - x)$ exists and equal to $f(x)$ *a.e.* That is $F'(x) = f(x)$ *a.e.*~~

By Theorem

3/22 - 3.1

Lemma: If f bounded and measurable, suppose

$$F(x) = F(a) + \int_{(a,x)} f\,dm \quad \text{then } F'(x) = f(x) \text{ a.e.}$$

Pf: Let $|f(x)| \le C$, $m(E) < \frac{\epsilon}{C} = \delta$

$$\int_E |f(x)|\,dm < \epsilon \Rightarrow F \ AC \Rightarrow F'(x)$$

exists a.e. Set $f_n(x) = \dfrac{F(x+\frac{1}{n}) - F(x)}{1/n}$

then $F'(x) = \lim_n f_n(x)$ a.e.

Also, $f_n(x) = n \int_{(x, x+1/n)} f\,dm \le n \int_{(x, x+1/n)} |f|\,dm \le C$

By BCT,

$$\int_a^c F'(x)\,dx = \lim_n \int_a^c f_n(x)\,dx =$$

$$\lim_n n\left[\int_{a+1/n}^{c+1/n} F(x) - \int_a^c F(x)\,dx\right] = \lim_n \left[n\int_c^{c+1/n} F\,dm - n\int_a^{a+1/n} F\,dm\right]$$

$= F(c) - F(a)$ since F cont.

3/22-3.2

Thm f integrable on $[a,b]$, suppose

$$F(x) = F(a) + \int_{(a,x)} f \, dm. \quad \text{Then } F'(x) = f(x) \text{ a.e.}$$

Pf: Set $f_n(x) = \begin{cases} f(x) & \text{if } f(x) \le n \\ n & \text{if } f(x) \ge n \end{cases}$

Then f_n bounded

$$F_n(x) = F(a) + \int_{(a,x)} f_n \, dm \Rightarrow F_n'(x) = f_n(x) \text{ a.e.}$$

Let $G_n(x) = \int_a^x (f - f_n)$ so $G_n(x)$ increasing

$\Rightarrow G_n'$ exists a.e.

$f - f_n \ge 0$ $\qquad F(x) = F_n(x) + G_n(x)$

$\Rightarrow F'(x)$ exists a.e., $F'(x) = F_n'(x) + G_n'(x) \ge f_n(x)$

$\Rightarrow F'(x) \ge f(x)$

$\Rightarrow \int_a^c F'(x) \ge \int_a^c f \, dm = F(c) - F(a)$

since increasing

But $\int_a^c F'(x) = \int_a^c F_n'(x) + \int_a^c G_n'(x) \le F_n(c) - F_n(a) + G_n(c) - G_n(a)$

$= F(c) - F(a)$

Proof of (c) $\Rightarrow$ (a): Suppose that F' exists $a.e.$ and $F' \in L^1$, and $F(x) - F(a) = \int_{[a,x]} F' dm \Rightarrow |F(b_i) - F(a_i)| \leq \int_{(a_i,b_i)} |f| dm$ where $f = F'$. Since $|f| \in L^1$, by Corollary 3.6, given $\epsilon > 0$, there exists $\delta > 0$ such that $m(E) < \delta \Rightarrow \int_E |f| dm$. So, take $E = \bigcup_{i=1}^{n} (a_i, b_i)$, then $m(E) = \sum_{i=1}^{n} b_i - a_i < \delta \Rightarrow \sum_{i=1}^{n} |F(b_i) - F(a_i)|$ $= \sum_{i=1}^{n} |\int_{(a_i,b_i)} F' dm| \leq \sum_{i=1}^{n} \int_{(a_i,b_i)} |F'| dm = \int_{\cup(a_i,b_i)} |F'| dm < \epsilon$. Thus, F is absolutely continuous on $[a, b]$.

(b) $\Rightarrow$ (a): Apply Thm 3.5

3.6 Applications of Absolute Continuity

Definition: A function $F : [a, b] \to \mathbb{R}$ is **Lipschitz** with a constant M provided that for all $x, y \in [a, b]$, $|F(y) - F(x)| \leq M|y - x|$.

Proposition: A function $F : [a, b] \to \mathbb{R}$ is Lipschitz with a constant M if and only if F is absolutely continuous and $|F'(x)| \leq M$ *a.e.*
In particular, if F is Lipschitz, then $F'(x)$ exists *a.e.* and $F(x) - F(a) = \int_{[a,x]} F' dm$.

Proof ($\Leftarrow$): Given $\epsilon > 0$, let $\delta = \epsilon/M$. Then, for any (a_1, b_1), (a_2, b_2), $\cdots, (a_n, b_n)$ disjoint intervals with $\sum_{i=1}^{n}(b_i - a_i) < \delta \Rightarrow$ $\sum_{i=1}^{n}|F(b_i) - F(a_i)| \leq \sum_{i=1}^{n} M|b_i - a_i| < M \cdot \delta = M \cdot \epsilon/M = \epsilon$.
Thus, F is absolutely continuous. This implies that $F'(x)$ exists *a.e.*, and at any point where $F'(x)$ exists, $|F'(x)| = \lim_{y \to x}|(F(y) - F(x))/(y - x)|$. But, by Lipschitz condition, $|F(y) - F(x)| \leq M|y - x|$. Thus, $|F'(x)| \leq M$.

Proof ($\Rightarrow$): Suppose that F is absolutely continuous and $|F'(x)| \leq M$. Then, $|F(y) - F(x)| \leq |\int_{[x,y]} F' dm| \leq \int_{[x,y]} |F'| dm \leq M \cdot m([x, y]) = M|y - x|$. Thus, F is Lipschitz with constant M.

Definition: A function $F : (a, b) \to \mathbb{R}$ is **convex** if for all $a < s < t < b$ and $0 \leq \lambda \leq 1$, $F(\lambda s + (1 - \lambda)t) \leq \lambda F(s) + (1 - \lambda)F(t)$.

Lemma 1: Let $F : (a, b) \to \mathbb{R}$ be convex, and $a < s < r \leq t < b$, then $(F(r) - F(s))/(r - s) \leq (F(t) - F(s))/(t - s)$.

Proof: Write $r = \lambda s + (1 - \lambda)t \Rightarrow r - t = \lambda(s - t) \Rightarrow \lambda = (r - t)/(s - t) = (t - r)/(t - s) \Rightarrow 1 - \lambda = 1 - (t - r)/(t - s) = [(t - s) - (t - r)]/(t - s) = (r - s)/(t - s)$. Thus, $(F(r) - F(s))/(r - s) \leq [\lambda F(s) + (1 - \lambda)F(t)]/(r - s) = [-(1 - \lambda)F(s) + (1 - \lambda)F(t)]/(r - s) = [(F(t) - F(s))(1 - \lambda)]/(r - s) = [(F(t) - F(s)) \cdot (r - s)/(t - s)]/(r - s) = (F(t) - F(s))/(t - s)$.

Lemma 2: Let $a_1 < b_1 \leq a_2 < b_2$. Then, $(F(b_1) - F(a_1))/(b_1 - a_1) \leq (F(b_2) - F(a_2))/(b_2 - a_2)$.