~~HW 1~~ Def'n: Given sets $\{E_n\}$ we say that $x \in E_n$ infinitely often (i.o.) if $\{n : x \in E_n\}$ is infinite.

We say that $x \in E_n$ almost always (a.a.) if $\mathbb{N} \setminus \{n : x \in E_n\}$ is finite

HW 1 Let $\mathcal{A}$ be a σ-alg on X, let $E_n \in \mathcal{A}$, $n \in \mathbb{N}$. Prove that

i) $A = \{x : x \in E_n \text{ i.o.}\} \in \mathcal{A}$

ii) $B = \{x : x \in E_n \text{ a.a.}\} \in \mathcal{A}$

Def'n Given a set $E \subseteq X$, we write χ_E for the function $\chi_E(x) = \begin{cases} 1, & x \in E \\ 0, & x \notin E \end{cases}$

HW 2 For A, B as in HW 1. Prove that

$\chi_A(x) = \limsup \chi_{E_n}(x)$

$\chi_B(x) = \liminf \chi_{E_n}(x)$. ~~write~~

Examples:

(1) $\mathcal{P}(X)$ is a σ-algebra.

(2) Let X be an infinite set, and let $\mathcal{A} = \{A \subseteq X : \text{either } A \text{ is finite or co-finite (that is, } A^c \text{ is finite.)}\}$ Then, $A_n = \{x_n\}$ is finite for each n, but $\bigcup_{n=1}^{\infty} A_n$ is not finite. By carefully choosing the x_ns, we can guarantee that $\bigcup_{n=1}^{\infty} A_n$ is also not co-finite.
Thus, $\mathcal{A}$ is not a σ-algebra.

(3) Let X be an uncountable set, and let $\mathcal{A} = \{A \subseteq X : A \text{ is at most countable or at most co-countable (that is } A^c \text{ is at most countable.)}\}$ Then, $\mathcal{A}$ is a σ-algebra.

Proposition: Let X be a nonempty set. Suppose that $\mathcal{A}_\alpha \subseteq \mathcal{P}(X)$ where each $\mathcal{A}_\alpha$ is a σ-algebra. Let $\bigcap_\alpha \mathcal{A}_\alpha \equiv \{A \subseteq X : A \in \mathcal{A}_\alpha \text{ for all } \alpha\}$. Then, $\bigcap_\alpha \mathcal{A}_\alpha$ is also a σ-algebra.

Proof: If $A \in \bigcap_\alpha \mathcal{A}_\alpha$, then $A \in \mathcal{A}_\alpha$ for all α. Since each $\mathcal{A}_\alpha$ is a σ-algebra, $A \in \mathcal{A}_\alpha$ for all $\alpha \Rightarrow A^c \in \mathcal{A}_\alpha$ for all $\alpha \Rightarrow A^c \in \bigcap_\alpha \mathcal{A}_\alpha$.
Now, $A_n \in \bigcap_\alpha \mathcal{A}_\alpha$ for all $n \Rightarrow A_n \in \mathcal{A}_\alpha$ for all α and all $n \Rightarrow$
$\bigcup_{n=1}^{\infty} A_n \in \mathcal{A}_\alpha$ for all $\alpha \Rightarrow \bigcup_{n=1}^{\infty} A_n \in \bigcap_\alpha \mathcal{A}_\alpha$.

Definition: Let X be a nonempty set, $\mathcal{E} \neq \emptyset$ and $\mathcal{E} \subseteq \mathcal{P}(X)$. Suppose that $\mathcal{A}_\alpha$ denotes the collection of all σ-algebras with property that $\mathcal{E} \subseteq \mathcal{A}_\alpha$ for all α. Then, $\bigcap_\alpha \mathcal{A}_\alpha$ is called the <u>σ-algebra generated by $\mathcal{E}$</u>, denoted by $\mathcal{M}(\mathcal{E})$.

<u>Note</u>: $\mathcal{P}(X) \in \mathcal{A}_\alpha$ because $\mathcal{P}(X)$ is a σ-algebra.
By the proposition above, $\bigcap_\alpha \mathcal{A}_\alpha$ is a σ-algebra and $\mathcal{E} \subseteq \bigcap_\alpha \mathcal{A}_\alpha$.
$\mathcal{M}(\mathcal{E})$ is the smallest σ-algebra containing $\mathcal{E}$, that is if $\mathcal{A}$ is a σ-algebra and $\mathcal{E} \subseteq \mathcal{A}$, then $\mathcal{M}(\mathcal{E}) \subseteq \mathcal{A}$.
If $\mathcal{E}$ is a σ-algebra, then $\mathcal{M}(\mathcal{E}) = \mathcal{E}$. $\mathcal{E}_\sigma, \mathcal{E}_\delta, \mathcal{E}_{\sigma\delta}$, etc.

Definition: If X is a metric or topological space, then the σ-algebra generated by the open subsets of X is called the **σ-algebra of Borel subsets of X**, denoted by $\mathcal{B}(X)$.

<u>Lemma</u> If $\mathcal{A}$ is a σ-algebra, $\mathcal{E} \subseteq \mathcal{A}$, then $\mathcal{M}(\mathcal{E}) \subseteq \mathcal{A}$

Proposition 1.2: The following collections of sets are equal:

(1) $\mathcal{B}_1 = \mathcal{B}(\mathbb{R})$

(2) $\mathcal{B}_2 =$ the σ-algebra generated by open intervals (a, b) where $a < b$.

(3) $\mathcal{B}_3 =$ the σ-algebra generated by closed intervals $[a, b]$ where $a < b$.

(4) $\mathcal{B}_4 =$ the σ-algebra generated by half-open intervals $(a, b] = \{x : a < x \leq b\}$.

(5) $\mathcal{B}_5 =$ the σ-algebra generated by $(a, +\infty)$ and $(-\infty, a)$.

(6) $\mathcal{B}_6 =$ the σ-algebra generated by $[a, +\infty)$ and $(-\infty, a]$.

Proof: [Show $\mathcal{B}_2 \subseteq \mathcal{B}_1$] Each (a, b) is open $\Rightarrow (a, b) \in \mathcal{B}(\mathbb{R}) = \mathcal{B}_1$. Thus, by Lemma above, $\mathcal{B}_2 \subseteq \mathcal{B}_1$.

[Show $\mathcal{B}_3 \subseteq \mathcal{B}_2$] $[a, b] = \bigcap_{n=1}^{\infty} (a - 1/n, b + 1/n) \in \mathcal{B}_2$. Thus, by Lemma above, $\mathcal{B}_3 \subseteq \mathcal{B}_2$.

[Show $\mathcal{B}_4 \subseteq \mathcal{B}_3$] $(a, b] = \bigcup_{n=1}^{\infty} [a + 1/n, b] \in \mathcal{B}_3$. Thus, by Lemma above, $\mathcal{B}_4 \subseteq \mathcal{B}_3$.

[Show $\mathcal{B}_5 \subseteq \mathcal{B}_4$] $(a, +\infty) = \bigcup_{n=1}^{\infty} (a, n] \in \mathcal{B}_4$ and $(-\infty, a) = \bigcup_{n=1}^{\infty} (-n, a - 1/n) \in \mathcal{B}_4$. Thus, by Lemma above $\mathcal{B}_5 \subseteq \mathcal{B}_4$.

[Show $\mathcal{B}_6 \subseteq \mathcal{B}_5$] $[a, +\infty) = \bigcap_{n=1}^{\infty} (a - 1/n, +\infty) \in \mathcal{B}_5$ and $(-\infty, a] = \bigcap_{n=1}^{\infty} (-\infty, a + 1/n) \in \mathcal{B}_5$. Thus, by Lemma above $\mathcal{B}_6 \subseteq \mathcal{B}_5$.

[Show $\mathcal{B}_1 \subseteq \mathcal{B}_6$] If $a < b$, then $[a, b] = (-\infty, b] \cap [a, +\infty) \in \mathcal{B}_6$ and $(a, b) = \bigcup_{n=1}^{\infty} [a + 1/n, b - 1/n] \in \mathcal{B}_6$.

Let $O \subseteq \mathbb{R}$ be any open set. Then, $O = \bigcup\{(a, b) : (a, b) \subseteq O$ where $a, b \in \mathbb{Q}\} \in \mathcal{B}_6$. Thus, by Lemma above $\mathcal{B}_1 \subseteq \mathcal{B}_6$.

Note that a similar proof shows that $\mathcal{B}(\mathbb{R}^2)$ is the σ-algebra generated by open rectangles $(a, b) \times (c, d)$.

The Extended Reals

$$\mathbb{R}_e = \mathbb{R} \cup \{+\infty\} \cup \{-\infty\}$$

ORDER $\quad -\infty < r < +\infty \quad \forall\, r \in \mathbb{R}$

Addition $\quad +\infty + r = +\infty, \quad -\infty + r = -\infty$

$+\infty + \infty = +\infty, \quad -\infty + (-\infty) = -\infty$

$+\infty + (-\infty)$ not defined

Multiplication $\quad r > 0, \quad r \cdot (+\infty) = +\infty$

$r \cdot (-\infty) = -\infty$

$r < 0, \quad r \cdot (+\infty) = -\infty, \quad r(-\infty) = +\infty$

$(+\infty)(-\infty) = -\infty,$ etc.

$0 \cdot (+\infty), \; 0 \cdot (-\infty)$ not defined.

1.3 Measures

Definition: Let X be a set and $\mathcal{M}$ be a σ-algebra of subsets of X.
A function $\mu : \mathcal{M} \to [0, +\infty]$ is a **measure** provided that:

(i) $\mu(\emptyset) = 0$

(ii) If $\{E_n\} \subseteq \mathcal{M}$, countable collection of disjoint sets in $\mathcal{M}$, then $\mu(\bigcup\limits_{n=1}^{\infty} E_n) = \sum\limits_{n=1}^{\infty} \mu(E_n)$.

Note that if μ is a measure and $E_1, E_2, \ldots, E_N \in \mathcal{M}$, then $\mu(\bigcup\limits_{n=1}^{N} E_n) = \sum\limits_{n=1}^{N} \mu(E_n)$. To see this, take $E_n = \emptyset$ for all $n > N$.

$(X, \mathcal{M})$ is called a **measurable space**, and $(X, \mathcal{M}, \mu)$ is called a **measure space**. When $\mu(X) = 1$ we call $(X, \mathcal{M}, \mu)$ a probability space, elements of $\mathcal{M}$ are called events.

Examples:

(i) Let X be any nonempty set and $\mathcal{M} = \mathcal{P}(X)$.

$$\mu(E) = \begin{cases} card(E) & \text{if } E \text{ is finite} \\ +\infty & \text{if } E \text{ is infinite} \end{cases}$$

Then, μ is a measure on $\mathcal{M}$, and is called a **counting measure**.

skip

(i') Let X be any nonempty set and $\mathcal{M} = \mathcal{P}(X)$.
Let $f : X \to [0, +\infty)$, and define $\mu(E) = \sum\limits_{x \in E} f(x)$.
Then, μ is a measure, and is called **weighted counting measure**. Note that we get (i) by letting $f(x) = 1$ for all x.

(ii) Let X be an uncountable set, and $\mathcal{M} = \{A : A$ is countable or co-countable, that is A^c is countable.$\}$

$$\text{Define } \mu_1(E) = \begin{cases} 0 & \text{if } E \text{ is countable} \\ 1 & \text{if } E \text{ is uncountable} \end{cases}$$

Then, μ_1 is a measure.

(ii') Let X be an uncountable set, and $\mathcal{M} = \{A : A$ is countable or co-countable, that is A^c is countable.$\}$

$$\text{Define } \mu_2(E) = \begin{cases} 0 & \text{if } E \text{ is countable} \\ +\infty & \text{if } E \text{ is uncountable} \end{cases}$$

Then, μ_2 is a measure.

(iii) Let X be an infinite set and $\mathcal{M} = \mathcal{P}(X)$.

$$\text{Define } \mu(E) = \begin{cases} 0 & \text{if } E \text{ is finite} \\ +\infty & \text{if } E \text{ is infinite} \end{cases}$$

Then, μ is <u>not</u> a measure, but it is finitely additive.

Examples continued:

~~(iii) Let X be an infinite set and $\mathcal{M} = \mathcal{P}(X)$.~~

~~Define $\mu_3(E) = \begin{cases} card(E) & \text{if } E \text{ is finite} \\ +\infty & \text{if } E \text{ is infinite} \end{cases}$~~

~~Then, μ_3 is a measure.~~

Properties of Measures:

- μ is **finite** if $\mu(E) < +\infty$ for all $E \in \mathcal{M}$.
- μ is **σ-finite** if $X = \bigcup_{n=1}^{\infty} E_n$, $E_n \in \mathcal{M}$ and $\mu(E_n) < +\infty$ for all n.
- μ is **semifinite** if $\mu(E) = +\infty$, then there exists $F \subseteq E$, $F \in \mathcal{M}$ and $0 < \mu(F) < +\infty$.

Consider the measures μ, μ_1, μ_2, and μ_3 in the above examples, and note that:
μ is finite if and only if X is finite.
μ is σ-finite if and only if X is countable.
μ is always semifinite.
~~If μ_1 is finite, then~~ μ_1 is σ-finite.
μ_2 is not finite.
μ_2 is not σ-finite.
μ_2 is not semifinite.
~~μ_3 is not finite~~.
~~μ_3 is not σ-finite~~.
~~μ_3 is semifinite~~.

Theorem 1.8: Let $(X, \mathcal{M}, \mu)$ be a measure space. Then:

(a) (Monotonicity) If $E, F \in \mathcal{M}$ and $E \subseteq F$, then $\mu(E) \leq \mu(F)$.

(b) (Subadditivity) If $E_n \in \mathcal{M}$, then $\mu(\bigcup_{n=1}^{\infty} E_n) \leq \sum_{n=1}^{\infty} \mu(E_n)$.

(c) (Continuity from below) If $E_n \in \mathcal{M}$ and $E_1 \subseteq E_2 \subseteq \ldots$, then $\mu(\bigcup_{n=1}^{\infty} E_n) = \lim_{n\to\infty} \mu(E_n)$.

(d) (Continuity from above) If $E_n \in \mathcal{M}$ and $E_1 \supseteq E_2 \supseteq \ldots$, and there exists n_0 such that $\mu(E_{n_0}) < +\infty$, then $\mu(\bigcap_{n=1}^{\infty} E_n) = \lim_{n\to\infty} \mu(E_n)$.

The following example shows that why we need a stronger condition for (d): Consider $(\mathbb{N}, \mathcal{P}(\mathbb{N}), \mu)$ where μ is the counting measure, and let $E_1 = \{1, 2, 3, \ldots\}$, $E_2 = \{2, 3, 4, \ldots\}, \ldots,$ $E_n = \{n, n+1, n+2, \ldots\}$. Notice that $E_1 \supseteq E_2 \supseteq \ldots$, and $\lim_{n\to\infty} \mu(E_n) = +\infty$, $\bigcap_{n=1}^{\infty} E_n = \emptyset$ and $\mu(\bigcap_{n=1}^{\infty} E_n) = 0$. Thus, $\lim_{n\to\infty} \mu(E_n) \neq \mu(\bigcap_{n=1}^{\infty} E_n)$.

Proof (a): Let $E \subseteq F$. Then $F = E \,\dot{\cup}\, (F \backslash E)$, and so $\mu(F) = \mu(E) + \mu(F\backslash E) \geq \mu(E)$.

Proof (b): Let $F_1 = E_1, F_2 = E_2 \backslash E_1, \ldots, F_n = E_n \backslash (\bigcup_{j=1}^{n-1} E_j)$. Then, F_is are disjoint and $F_1 \cup F_2 \cup \ldots \cup F_n = E_1 \cup E_2 \cup \ldots \cup E_n$. So, $\mu(\bigcup_{j=1}^{\infty} E_j) = \mu(\bigcup_{j=1}^{\infty} F_j) = \sum_{j=1}^{\infty} \mu(F_j) \leq \sum_{j=1}^{\infty} \mu(E_j)$ by (a) because $F_j \subseteq E_j$ for all j.

Proof (c): Let $E_0 = \emptyset$, $F_j = E_j \backslash E_{j-1}$. Then, F_js are disjoint, $\bigcup_{j=1}^{n} F_j = E_n$, and $\bigcup_{j=1}^{\infty} F_j = \bigcup_{j=1}^{\infty} E_j$. Thus, $\mu(\bigcup_{j=1}^{\infty} E_j) = \mu(\bigcup_{j=1}^{\infty} F_j) = \sum_{j=1}^{\infty} \mu(F_j) = \lim_{n\to\infty} \sum_{j=1}^{n} \mu(F_j) = \lim_{n\to\infty} \mu(E_n)$.

Proof (d): Since E_ns are decreasing, $\bigcap_{n=1}^{\infty} E_n = \bigcap_{n=n_0}^{\infty} E_n$. Let $F_j = E_{n_0} \backslash E_j$ where $j \geq n_0$. Then, $F_j = E_{n_0} \backslash E_j$ where $j \geq n_0$. Then, $F_{n_0} = \emptyset \subseteq F_{n_0+1} \subseteq F_{n_0+2} \subseteq \ldots$, and $E_j \,\dot{\cup}\, F_j = E_{n_0}$. So, $\mu(E_{n_0}) = \mu(E_j) + \mu(F_j)$, and $\bigcup_{n=n_0}^{\infty} F_n = E_{n_0} \backslash \bigcap_{j=1}^{\infty} E_j$.

Now by (c), $\mu(E_{n_0} \backslash \bigcap_{j=1}^{\infty} E_j) = \mu(\bigcup_{n=n_0}^{\infty} F_n) = \lim_{n\to\infty} \mu(F_n) = \lim_{n\to\infty} (\mu(E_{n_0}) - \mu(E_n)) = \mu(E_{n_0}) - \lim_{n\to\infty} \mu(E_n)$. On the other hand $\mu(E_{n_0} \backslash \bigcap_{j=1}^{\infty} E_j) = \mu(E_{n_0}) - \mu(\bigcap_{j=1}^{\infty} E_j)$. Thus, $\mu(\bigcap_{j=1}^{\infty} E_j) = \lim_{n\to\infty} \mu(E_n)$.

HW3 Let $(X, \mathcal{M}, \mu)$ measure, $E_n \in \mathcal{M}$, $A = \{x : x \in E_n \text{ i.o.}\}$
Prove that if $\sum_{n=1}^{\infty} \mu(E_n) < +\infty$, then $\mu(A) = 0$.