HW

HW 16: Let $(X, \mathcal{M}, \mu)$, $(X, \mathcal{M}, \nu)$ be σ-finite measures. Let $\lambda = \mu + \nu$

Prove: i) $\mu \ll \lambda$, $\nu \ll \lambda$

ii) $\frac{d\mu}{d\lambda} + \frac{d\nu}{d\lambda} = 1$ a.e. λ

iii) $\nu \ll \mu \iff \lambda(\{x : \frac{d\mu}{d\lambda}(x) = 0\}) = 0$

HW 17: Let $E \subseteq \mathbb{R}$ be measurable. Set

$$D_E(x) = \lim_{r \to 0^+} \frac{m(E \cap (x-r, x+r))}{2r} \quad \text{when it exists}$$

Note that $0 \le D_E(x) \le 1$.

For each $0 \le t \le 1$, construct a measurable set E such that, $D_E(0) = t$.

The Derivates of f:

$$D^+ f(x) = \overline{\lim_{h \to 0^+}} \frac{f(x+h) - f(x)}{h}$$

$$D^- f(x) = \overline{\lim_{h \to 0^-}} \frac{f(x) - f(x\ h)}{h}$$

$$D_+ f(x) = \underline{\lim_{h \to 0^+}} \frac{f(x+h) - f(x)}{h}, \quad D_- f(x) = \underline{\lim_{x \to 0}} \frac{f(x) - f(x-h)}{h}$$

$D^+ f(x) \geq D_+ f(x)$, $D^- f(x) \geq D_- f(x)$

$f'(x) \in \mathbb{R}$ exists $\iff$ All four are equal and not $\pm \infty$.

Thm Let f be increasing ($x \leq y \Rightarrow f(x) \leq f(y)$) on $[a, b]$.
Then $f'(x)$ exists a.e. m, f' is measurable and

$$\int_a^b f'(x)\, dm \leq f(b) - f(a)$$

Pf: ~~Show~~ For each pair of derivates, show that the set of x where they are not equal is measure 0.

Only do $E = \{x : D^+ f(x) > D_- f(x)\}$. Fix $u < v$ rational

Let $E_{u,v} = \{x : D^+ f(x) > u > v > D_- f(x)\}$, $E = \underbrace{\bigcup_{u,v}}_{\text{countable}} E_{u,v}$

Enough to show $m^*(E_{u,v}) = 0$.

Let $s = m^*(E_{u,v})$. Fix $\epsilon > 0$, $E_{u,v} \subseteq \mathcal{O}$, open $m(\mathcal{O}) < s + \epsilon$

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For each $x \in E_{u,v}$ $\exists$ $[x-h, x] \subseteq \mathcal{O}$

$\ni$ $\dfrac{f(x) - f(x-h)}{h} < v$ Vitali cover.

$\exists$ $I_1, \dots, I_N$ $\ni$ $A = E_{u,v} \cap \left(\bigcup_{n=1}^{N} I_N \right)$

$E_{u,v} = A \,\dot{\cup}\, (E_{u,v} \setminus \bigcup I_n) \Rightarrow m^*(E_{u,v}) \le m^*(A) + \varepsilon$

$m^*(A) > S - \epsilon$

$$\sum_{n=1}^{N} f(x_n) - f(x_n - h_n) < v \sum h_n < v\, m(\mathcal{O}) < v(S+\epsilon)$$

$y \in A$, $\exists$ $(y, y+k) \subseteq I_j$ some $1 \le j \le N$

and with k small enough

$\Rightarrow$ $f(y+k) - f(y) > k \cdot u$

Vitali pick $J_1, \dots, J_M$ $\ni$ union contains

$m^*(A \setminus \bigcup J_i) < \varepsilon \Rightarrow m^*(A \cap (\bigcup J_i)) > S - 2\epsilon$

$$\sum_{1}^{M} f(y_i + k_i) - f(y_i) > u \sum k_i \ge u\,(m^*(\bigcup J_i)) > u(S - 2\epsilon)$$

$$\therefore\ u(S-2\epsilon) < \sum_{i=1}^{M} f(y_i + k_i) - f(y_i) \le \sum_{n=1}^{N} f(x_n) - f(x_n - h_n) < v(S+\epsilon)$$

$J_i \subseteq I_n$

True $\forall \varepsilon$ $\Longrightarrow$ $uS \le vS$; $u > v$ $\Rightarrow$ $S = 0$

$\therefore$ $g(x) = \lim_{h \to 0} \dfrac{f(x+h) - f(x)}{h}$ is defined a.e.

$g(x) = f'(x)$ where defined

Set $g_n(x) = \begin{cases} n\left(f(x+\frac{1}{n}) - f(x)\right) \\ \end{cases}$

where we set $f(y) = f(b)\ \forall\ y \geq b$, $f(y) = f(a)\ \forall\ y \leq a$

g_n measurable since f increasing $\Rightarrow f$ meas.

$g_n(x) \to g(x)$ a.e. $\to g$ meas.

Fatou

$$\int_a^b f'\,dm \leq \underline{\lim} \int_a^b g_n\,dm$$

$$= \underline{\lim}\ n\left[\int_{a+\frac{1}{n}}^{b+\frac{1}{n}} f\,dm - \int_a^b f\,dm\right] = \underline{\lim}\ n\left(\int_b^{b+\frac{1}{n}} f - \int_a^{a+1/n} f\right)$$

$$= \underline{\lim}\left\{n\left(\tfrac{1}{n} f(b)\right) - n\int_a^{a+1/n} f\right\}$$

$$\leq \underline{\lim}\ f(b) - \left[n\,\tfrac{1}{n} f(a)\right] = f(b) - f(a)$$

~~Next, let $H(x) = G(x) - F(x) \geq 0$. Then, $H(x) = 0$ except on a countable set $\Rightarrow G(x) = F(x)$ except on a countable set. [We need to show that H' exists and $H'(x) = 0$ $a.e.$ x.]~~

Definition: A function $F : [a, b] \to \mathbb{C}$ is said to be of **bounded variation** if

$$V_F([a,b]) = sup\{\sum_{j=1}^{n} |F(x_j) - F(x_{j-1})| : a = x_0 < x_1 < \cdots < x_n = b\}$$

over all partitions, is finite. We let $BV([a,b])$ denote the set of all functions on $[a,b]$ of bounded variation.
More generally, a function $F : \mathbb{R} \to \mathbb{C}$ is said to be of **bounded variation** if $V_F(\mathbb{R}) = sup\{\sum_{j=1}^{n} |F(x_j) - F(x_{j-1})| : x_0 < x_1 < \cdots < x_n\}$ over all partitions, is finite. We let BV denote the set of all such functions.
Let $T_F(x) = sup\{\sum_{j=1}^{n} |F(x_j) - F(x_{j-1})| : x_0 < x_1 < \cdots < x_n = x\}$ for $F \in BV$. Then, T_F is called the **total bounded variation** of F.
<u>Note</u> that $0 \leq T_F(x) \leq V_F(\mathbb{R}) < +\infty$. If $x < y$, then $T_F(x) \leq T_F(y)$.

Examples:

(1) Let $F(x) = x$. Then, $F \notin BV$, but $F \in BV([a,b])$, and for any $[a,b]$, $V_F([a,b]) = |b - a|$.

(2) Given $F \in BV([a,b])$, let $\widetilde{F}(x) = \begin{cases} F(a) & \text{if } x \leq a \\ F(x) & \text{if } a \leq x \leq b \\ F(b) & \text{if } b \leq x \end{cases}$

Then, $\widetilde{F} \in BV$, and $V_{\widetilde{F}}(\mathbb{R}) = V_F([a,b])$.

(3) Let $F : \mathbb{R} \to \mathbb{R}$ be increasing and bounded. Then, $\lim_{x \to +\infty} F(x) = F(+\infty)$ exists and $\lim_{x \to -\infty} F(x) = F(-\infty)$ exists, and $F \in BV$ and $V_F(\mathbb{R}) = F(+\infty) - F(-\infty)$.
[F is increasing, so $\sum_{j=1}^{n} |F(x_j) - F(x_{j-1})|$ is telescoping.]

(4) If $F, G \in BV$ and $a, b \in \mathbb{C}$, then $aF + bG \in BV$, and $V_{aF+bG}(\mathbb{R}) \leq |a|V_F(\mathbb{R}) + |b|V_G(\mathbb{R}) \Rightarrow BV$ is a vector space.

(5) If $h \in L^1(\mathbb{R})$, and set $F(x) = \int_{(-\infty,x)} h dm$, then $F \in BV$ and $V_F(\mathbb{R}) \leq \int_{\mathbb{R}} |h| dm$.

Proof: $\sum_{j=1}^{n} |F(x_j) - F(x_{j-1})| = \sum_{j=1}^{n} |\int_{[x_{j-1},x_j]} h dm| \leq \sum_{j=1}^{n} \int_{[x_{j-1},x_j]} |h| dm$
$= \int_{[x_0,x_n]} |h| dm \leq \int_{\mathbb{R}} |h| dm < +\infty$ since $h \in L^1(\mathbb{R})$.

Lemma 3.26: Let F be a real valued function and $F \in BV$. Then, $T_F + F$ and $T_F - F$ are both increasing.

Proof: Let $x < y$ and $x_0 < x_1 < \cdots < x_n = x < y$. Then, $\sum_{j=1}^{n} |F(x_j) - F(x_{j-1})| + |F(y) - F(x)| \le T_F(y)$. This implies that $sup\{\sum_{j=1}^{n} |F(x_j) - F(x_{j-1})| : x_0 < x_1 < \cdots < x_n = x\}$ $+ |F(y) - F(x)| \le T_F(y) \Rightarrow T_F(x) + |F(y) - F(x)| \le T_F(y)$. Thus, $T_F(x) + F(y) - F(x) \le T_F(y)$ and $T_F(x) + F(x) - F(y) \le T_F(y) \Rightarrow T_F(x) - F(x) \le T_F(y) - F(y)$ and $T_F(x) + F(x) \le T_F(y) + F(y) \Rightarrow T_F - F$ and $T_F + F$ are both increasing.

Theorem 3.27:

(a) $F \in BV \Leftrightarrow ReF \in BV$ and $ImF \in BV$.

(b) If $F : \mathbb{R} \to \mathbb{R}$, then $F \in BV \Leftrightarrow F = H_1 - H_2$ where each H_i is increasing and bounded.

(c) If $F \in BV$, then $F(x+) = \lim_{y \to x^+} F(y)$, $F(x-) = \lim_{y \to x^-} F(y)$, $\lim_{x \to +\infty} F(y) = F(+\infty)$ and $\lim_{x \to -\infty} F(y) = F(-\infty)$ all exist.

(d) If $F \in BV$, then the set of points where F is discontinuous is at most countable.

(e) If $F \in BV$ ~~and set $G(x) = F(x+)$, then G' and~~ then F' exist *a.e.* ~~and $G' = F'$ a.e.~~.

Proof of (a): $\sum_{j=1}^{n} |ReF(x_j) - ReF(x_{j-1})| \le \sum_{j=1}^{n} |F(x_j) - F(x_{j-1})|$ and $\sum_{j=1}^{n} |ImF(x_j) - ImF(x_{j-1})| \le \sum_{j=1}^{n} |F(x_j) - F(x_{j-1})|$. Thus, if $F \in BV$, then $ReF, ImF \in BV$. Also, $\sum_{j=1}^{n} |F(x_j) - F(x_{j-1})|$ $\le \sum_{j=1}^{n} |ReF(x_j) - ReF(x_{j-1})| + |ImF(x_j) - ImF(x_{j-1})|$. So, if $ReF, ImF \in BV$, then $F \in BV$.

Proof of (b): Let $H_1(x) = (T_F(x) + F(x))/2$ and $H_2(x) = (T_F(x) - F(x))/2$. Then, by Lemma 3.26, H_1 and H_2 are both increasing, and $H_1 - H_2 = F$. We know that $0 \le T_F(x) \le V_F(\mathbb{R}) < +\infty$, and thus, T_F is bounded. Fix $x_0 = 0$, then $|F(x) - F(0)| \le V_F(\mathbb{R}) \Rightarrow |F(x)| \le V_F(\mathbb{R}) + |F(0)| < +\infty$.

Thus, F is bounded $\Rightarrow T_F \pm F$ are bounded $\Rightarrow H_1$ and H_2 are both bounded.

Proof of (c): We know that all of these limits exist for bounded and increasing functions. Thus, they exist for H_1 and H_2, and so they exist for $F = H_1 - H_2$.

Proof of (d): We know that H_1 and H_2 are continuous except at a countable set by Theorem 3.23 $\Rightarrow F = H_1 - H_2$ is continuous except for at most countably many points.

Proof of (e): Argue as (d) using Theorem 3.23.

Recall that if $F \in BV$ and T_F is a total variation, then $F = (T_F + F)/2 - (T_F - F)/2$. This decomposition of F is called the **Jordan decomposition of F**, and $(T_F + F)/2$ is called the **positive variation of F** and $(T_F - F)/2$ is called the **negative variation of F**.

Definition: A function $F : \mathbb{R} \to \mathbb{C}$ is **absolutely continuous** if for all $\epsilon > 0$, there exists $\delta > 0$ such that for all n and for all disjoint intervals (a_1, b_1), $(a_2, b_2), \cdots, (a_n, b_n)$ whenever $\sum_{i=1}^{n}(b_i - a_i) < \delta$, $\sum_{i=1}^{n}|F(b_i) - F(a_i)| < \epsilon$. We say that $F : [a, b] \to \mathbb{C}$ is **absolutely continuous** if the above holds for all $(a_i, b_i) \subseteq [a, b]$. Note: absolute continuity $\Rightarrow$ uniform continuity

Proposition 3.32: Let $H : \mathbb{R} \to \mathbb{R}$ be increasing, bounded and right continuous. Then, $\mu_H \ll m$ if and only if H is absolutely continuous.

Proof ($\Rightarrow$): Let $\mu_H \ll m$. Then, by Theorem 3.8 (The Lebesgue-Radon-Nikodym Theorem), there exists $f \geq 0$ such that $H(b) - H(a) = \int_{(a,b)} f dm$ for all a and b. Now, since H is bounded, $\lim_{x \to +\infty} H(x) = H(+\infty) < +\infty$ and $\lim_{x \to -\infty} H(x) = H(-\infty) < -\infty \Rightarrow \int_{\mathbb{R}} f dm = H(+\infty) - H(-\infty) < +\infty \Rightarrow f \in L^1(m)$. By Theorem 3.5 or Corollary 3.6, if $f \in L^1(m)$ and given $\epsilon > 0$, there exists $\delta > 0$ such that $m(E) < \delta \Rightarrow |\int_E f dm| < \epsilon$. Now, given $(a_1, b_1), (a_2, b_2), \cdots, (a_n, b_n)$ disjoint and $\sum_{i=1}^{n}(b_i - a_i) < \delta$. Let $E = \bigcup_{i=1}^{n}(a_i, b_i)$, then $m(E) < \delta \Rightarrow |\int_E f dm| < \epsilon$. But, $\int_E f dm = \sum_{i=1}^{n} \int_{(a_i, b_i)} f dm = \sum_{i=1}^{n} H(b_i) - H(a_i) = \sum_{i=1}^{n} |H(b_i) - H(a_i)| < \epsilon$. Thus, H is absolutely continuous.