Examples of Lebesgue-Radon-Nikodym

1. Let $F(x) = \begin{cases} 0 & \text{if } x \leq 0 \\ x^2 & \text{if } 0 < x < 1 \\ x^2 + 1 & \text{if } 1 \leq x \end{cases}$

   Look at μ_F and m – Lebesgue measure. We want to write $\mu_F = \lambda + \rho$ where $\rho \ll m$ and $\lambda \perp m$.

   Take $F_1(x) = \begin{cases} 0 & \text{if } x \leq 0 \\ x^2 & \text{if } 0 < x \end{cases}$ and $F_2(x) = \begin{cases} 0 & \text{if } x < 1 \\ 1 & \text{if } 1 \leq x \end{cases}$

   Then, $F = F_1 + F_2$, and so $\mu_F = \mu_{F_1} + \mu_{F_2}$.

   [Show that $\mu_{F_2} \perp m$.]

   $\mu_{F_2}(E) = \begin{cases} 1 & \text{if } 1 \in E \\ 0 & \text{if } 1 \notin E \end{cases}$ That is, $\mu_{F_2} = \delta_{\{1\}}$, dirac delta, or point mass at 1. Let $A = \{1\}$ and $B = \mathbb{R} \backslash \{1\}$, then $\mu_{F_2}(B) = 0$ and $m(A) = 0$. Thus, $\mu_{F_2} \perp m$.

   [Show that $\mu_{F_1} \ll m$.]

   Let $h(x) = 2x \cdot \chi_{(0,+\infty]}$. Then, $\int_{(a,b]} h dm = \int_{(a,b]} 2x \cdot \chi_{(0,+\infty]} dm$. If $0 < a$, then $\int_{(a,b]} 2x \cdot \chi_{(0,+\infty]} dm = \int_a^b 2x dx = x^2]_a^b = b^2 - a^2 = F_1(b) - F_1(a)$. If $a < 0 < b$, then $\int_{(a,b]} 2x \cdot \chi_{(0,+\infty]} dm = \int_0^b 2x dx = x^2]_0^b = b^2 - 0^2 = F_1(b) - F_1(a)$. Similarly, in all cases, we can see that $\int_{(a,b]} 2x \cdot \chi_{(0,+\infty]} dm = F_1(b) - F_1(a)$. Thus, $\mu_{F_1}(E) = \int_E 2x \cdot \chi_{(0,+\infty]} dm$ for all E which is Borel and $d\mu_{F_1}/dm = h$, and $\mu_{F_1} \ll m$. Note that $d\mu_{F_1}/dm = h$ is called the Radon-Nikodym derivative,

Cantor Set)

2. Recall the Cantor function, $f : [0,1] \rightarrow [0,1]$, $f(0) = 1$, $f(1) = 1$, f is continuous and increasing, and f is constant on intervals the we "throw away" to get the Cantor set.

   Let $F(x) = \begin{cases} 0 & \text{if } x \leq 0 \\ f(x) & \text{if } 0 \leq x \leq 1 \\ 1 & \text{if } 1 \leq x \end{cases}$ Then, F is increasing and continuous. Let $E_n = (-n, +n]$. Then, $\mu_F(E_n) = F(+n) - F(-n) = 1 - 0 = 1$. Notice that $E_1 \subseteq E_2 \subseteq \cdots \subseteq \mathbb{R} = \bigcup_n E_n \Rightarrow \mu_F(\mathbb{R}) = \lim_n \mu_F(E_n) = 1$. But, $\mu_F([1, +\infty)) = 0$ and $\mu_F((-\infty, 0]) = 0$. Also, $\mu_F((1/3, 2/3]) = F(2/3) - F(1/3) = 1/2 - 1/2 = 0 \Rightarrow \mu_F((1/3, 2/3)) = 0$. Similarly, $\mu_F((1/9, 2/9) = 0$ and $\mu_F(7/9, 8/9)) = 0$. Thus, $\mu_F((-\infty, 0] \dot\cup [1, +\infty) \dot\cup (1/3, 2/3) \dot\cup \cdots) = 0 =$

$\mu_F(C^c)$ where C is the Cantor set. So, this implies that $\mu_F(C) = 1$. Also, $\mathbb{R} = C \dot{\cup} C^c$, $m(C) = 0$, $\mu_F(C^c) = 0 \Rightarrow \mu_F \perp m$. But, F is continuous on $\mathbb{R}$, so $\mu_F(\{x\}) = 0$ for all $x \in \mathbb{R}$. That is all the measures live in the Cantor set.

Further results of Radon-Nikodym

Let μ and ν be σ-finite measures on $(X, \mathcal{M})$. If $\nu \ll \mu$, then there exists f such that $\nu(E) = \int_E f d\mu$. If we have g such that $\nu(E) = \int_E g d\mu$ for all $E \in \mathcal{M}$, then $f = g$ *a.e.* with respect μ.
Any such function is called $d\nu/d\mu$, which is a equivalence class of functions.
If $\nu_1 \ll \mu$ and $\nu_2 \ll \mu$, and let $\nu = \nu_1 + \nu_2$, then $\nu \ll \mu$, and $\nu(E) = \nu_1(E) + \nu_2(E) = \int_E (d\nu_1/d\mu)\, d\mu + \int_E (d\nu_2/d\mu)\, d\mu = \int_E (d\nu_1/d\mu + d\nu_2/d\mu) d\mu \Rightarrow d\nu/d\mu = d\nu_1/d\mu + d\nu_2/d\mu \Rightarrow d(\nu_1 + \nu_2)/d\mu = d\nu_1/d\mu + d\nu_2/d\mu \; \mu \; a.e.$

Proposition 3.9: Let ν, μ and λ be σ-finite measures on $(X, \mathcal{M})$. If $\lambda \ll \nu$ and $\nu \ll \mu$, then $\lambda \ll \mu$ and $d\lambda/d\mu = d\lambda/d\nu \cdot d\nu/d\mu \; \mu \; a.e.$

Proof: Let ν, μ and λ be σ-finite measures on $(X, \mathcal{M})$. Suppose that $\lambda \ll \nu$ and $\nu \ll \mu$. Then, if $\mu(E) = 0$, then $\nu(0) \Rightarrow \lambda(E) = 0$. Thus, $\lambda \ll \mu$. By Exercise 14 in page 63, if $\nu(E) = \int_E f d\nu$, then for $g \geq 0$, $\int_X g d\nu = \int_X g f d\mu$. So, $\lambda(E) = \int_E (d\lambda/d\nu) d\nu = \int_X (\chi_E d\lambda/d\nu) d\nu = \int_X \chi_E (d\lambda/d\nu \cdot d\nu/d\mu) d\mu = \int_E (d\lambda/d\nu \cdot d\nu/d\mu) d\mu \Rightarrow d\lambda/d\mu = d\lambda/d\nu \cdot d\nu/d\mu$.

Corollary 3.10: Let λ and μ be σ-finite. suppose that $\lambda \ll \mu$ and $\mu \ll \lambda$, then $d\lambda/d\mu \cdot d\mu/d\lambda = 1 \; \mu \; a.e. = \lambda \; a.e.$

Proof: Let λ and μ be σ-finite. suppose that $\lambda \ll \mu$ and $\mu \ll \lambda \Rightarrow \lambda \ll \lambda$ and $1 = d\lambda/d\lambda = d\lambda/d\mu \cdot d\mu/d\lambda$ by Theorem 3.9.

Proposition 3.11: Let $\mu_1, \mu_2, \cdots, \mu_n$ be σ-finite measures, and suppose that $\mu = \mu_1 + \mu_2 + \cdots + \mu_n$. Then, $\mu_i \ll \mu \Rightarrow \mu_i(E) = \int_E (d\mu_i/d\mu) d\mu$ and $d\mu_1/d\mu + d\mu_2/d\mu + \cdots + d\mu_n/d\mu = 1 \; \mu \; a.e.$

Differentiation and Integration

What functions have the property that $f'(x)$ exists a.e. m, ~~and~~ is measurable and

$$\int_a^b f'(x)\,dm = f(b) - f(a) \quad ?$$

<u>Example</u> f Cantor function $f'(x) = 0 \ \forall x \notin C$

$\therefore \int_0^1 f'\,dm = 0$ but $f(1) = 1$, $f(0) = 0$.

First look at ~~monotone~~ increasing or decreasing functions

<u>Def'n</u> $E \subseteq \mathbb{R}$, $\mathcal{I}$ - collection of intervals. We say the $\mathcal{I}$ covers E in the sense of Vitali if given $x \in E$, $\epsilon > 0$ $\exists\, I \in \mathcal{I}$, $x \in I$ and $m(I) < \varepsilon$

<u>Lemma (Vitali)</u> $m^*(E) < +\infty$, $\mathcal{I}$ cover E in sense of Vitali. Then given $\epsilon > 0$ $\exists \{I_1, \ldots, I_N\} \subseteq \mathcal{I}$ disjoint $\ni m^*\left(E \setminus \bigcup_{i=1}^{N} I_i\right) < \epsilon$

<u>Pf</u> WLOG assume each interval closed $E \subseteq \mathcal{O}$ - open, $m(\mathcal{O}) < +\infty$. WLOG can assume each $I \subseteq \mathcal{O}$. Define $\{I_n\}$ inductively. Suppose $\{I_1, \ldots, I_n\}$ chosen. Let $k_n = \sup\{\ell(I) : I \in \mathcal{I},\ I \cap I_j = \emptyset\}$. $k_n \leq m(\mathcal{O}) < +\infty$

Unless $E \subseteq I_1 \cup \cdots \cup I_n$ then $\exists\, I_{n+1} \in \mathcal{I}$, $\ell(I_{n+1}) > \frac{k_n}{2}$

and I_{n+1} disjoint from $I_1 \cup \dots \cup I_n$

$\Rightarrow m(\cup I_n) = \sum m(I_n) \le m(O) < +\infty$

$\therefore \exists N \ni \sum_{N+1}^{\infty} m(I_n) < \epsilon/5$

$R = E \setminus \bigcup_{n=1}^{N} I_n$ <u>Claim</u> $m^*(R) < \epsilon$

Let $x \in R$ since $x \notin \bigcup_{i=1}^{N} I_n$ - closed $\exists$ I - o

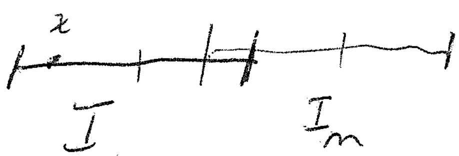
$x \in I$, $I \cap (\bigcup_{1}^{N} I_n) = \emptyset$

$\cancel{\Rightarrow \ell(I) \le k_N \le 2\ell(I_{N+1})}$

If $I \cap (\bigcup_{n=1}^{m} I_n) = \emptyset \Rightarrow \ell(I) \le k_m \to 0$

$\therefore \exists$ smallest $n \ni I \cap I_n \ne \emptyset$, and $n > N$

$I \cap (\bigcup_{i=1}^{n-1} I_i) \Rightarrow \ell(I) \le k_{n-1} \le 2\ell(I_n)$



$d(x, \text{midpoint of } I_n)$

$\le \ell(I) + \frac{1}{2}\ell(I_n) \le \frac{5}{2}\ell(I_n)$

Let J_n - interval same midpoint as I_n, 5 times the length

$\Rightarrow x \in J_n \Rightarrow R \subseteq \bigcup_{N+1}^{\infty} J_n$

$\Rightarrow m^*(R) \le \sum_{N+1}^{\infty} m(J_n) = 5 \sum_{N+1}^{\infty} m(I_n) < \epsilon$