HW 13: Let $E \subseteq [0,1] \times [0,1]$, $E \in \mathcal{B}(\mathbb{R}) \otimes \mathcal{B}(\mathbb{R})$

~~Def~~ and let m denote Lebesgue measure

If $m(E_x) \le 1/2$ for almost all $x \in [0,1]$,

then prove that $m(\{y : m(E^y) = 1\}) \le 1/2$.

HW 14 Let $f \in \mathcal{L}^1(\mathbb{R}, \mathcal{B}(\mathbb{R}), m)$, m Lebesgue measure. Set for $h > 0$

$$\phi_h(x) = \int_{x-h}^{x+h} f(t)\, dm(t).$$

Prove that ϕ_h is $\mathcal{B}(\mathbb{R})$-measurable,

$\phi_h \in \mathcal{L}^1(\mathbb{R}, \mathcal{B}(\mathbb{R}), m)$ and $\int_{\mathbb{R}} |\phi_h(x)|\, dm(x) \le 2h \int_{\mathbb{R}} |f|\, dm$

You will need to use properties of m, like translation invariance,

HW 15 Let $(X, \mathcal{M}, v)$ be a finite signed measure v^+, v^- Jordan decomposition and suppose that $v = v_1 - v_2$ where $(X, \mathcal{M}, v_i)$ are finite measures. Prove that $\forall\, E \in \mathcal{M}$

$v_1(E) \ge v^+(E)$ and $v_2(E) \ge v^-(E)$.

3.2 Radon-Nikodym and Lebesgue

Definition: Let ν be a signed measure and μ be a positive measure. We say that ν is **absolutely continuous with respect to** μ, denoted by $\nu \ll \mu$, if $\mu(E) = 0$, then $\nu(E) = 0$ for all $E \in \mathcal{M}$.

Example: $\nu(E) = \int_E f d\mu \Rightarrow \nu \ll \mu$

<u>Lemma</u>: ν finite signed, μ positive. $\nu \ll \mu \Leftrightarrow \nu^+ \ll \mu$, $\nu^- \ll \mu \Leftrightarrow |\nu| \ll \mu$

Theorem 3.5: Suppose that ν is a finite signed measure, and μ is a positive measure on $(X, \mathcal{M})$. Then, $\nu \ll \mu \Leftrightarrow$ for all $\epsilon > 0$, there exists $\delta > 0$ such that $|\nu(E)| < \epsilon$ whenever $\mu(E) < \delta$.

~~Remark: $\nu \ll \mu \Rightarrow \nu^+ \ll \mu$ and $\nu^- \ll \mu \Leftrightarrow |\nu| \ll \mu$~~

Proof: Without loss of generality, we can replace ν by $|\nu|$. So, we might as well do the case that ν is positive.
($\Leftarrow$) If $\mu(E) = 0$, take $\epsilon = 1/n$, then there exist $\delta_n > 0$ such that $\mu(E) < \delta_n$ for all $n \Rightarrow \nu(E) < 1/n$, and $\nu(E) = 0$. Thus, $\nu \ll \mu$.
($\Rightarrow$) Suppose that ϵ-δ condition fails. There exists $\epsilon_0 > 0$ for which no $\delta > 0$ can be found. This implies that for $\delta = 1/2^n$, there exists P_n such that $\mu(P_n) < 1/2^n$, but $\nu(P_n) \geq \epsilon_0$. Now, Let $F_k = \bigcup_{n=k}^{\infty} P_n$, and $F = \bigcap_{k=1}^{\infty} F_k$. Then, $\mu(F_k) \leq \sum_{n=k}^{\infty} \mu(P_n) \leq \sum_{n=k}^{\infty} 1/2^n = 1/2^{k-1} \Rightarrow \mu(F) = 0$. But, $\nu(F_k) \geq \nu(P_k) \geq \epsilon_0 \Rightarrow \nu(F) = \lim_k \nu(F_k) \geq \epsilon_0$. Thus, ν is not absolutely continuous with respect μ. Therefore, if $\nu \ll \mu$, then for all $\epsilon > 0$, there exists $\delta > 0$ such that $|\nu(E)| < \epsilon$ whenever $\mu(E) < \delta$.

Corollary 3.6: Let $f \in \mathcal{L}^1(\mu)$, then for all $\epsilon > 0$, there exists $\delta > 0$ such that $\mu(E) > 0$ implies that $|\int_E f d\mu| < \epsilon$.

Proof: Let $f \in \mathcal{L}^1(\mu)$. Let $\nu(E) = \int_E f d\mu$, then it is finite and $\nu \ll \mu$. Thus, by Theorem 3.5, for all $\epsilon > 0$, there exists $\delta > 0$ such that $\mu(E) > 0$ implies that $|\int_E f d\mu| = |\nu(E)| < \epsilon$.

Lemma 3.7: Let μ and ν be finite measures on $(X, \mathcal{M})$. Then, either $\nu \perp \mu$ or there exists $\epsilon > 0$ and $E \in \mathcal{M}$ with $\mu(E) > 0$ and $\nu \geq \epsilon\mu$ on E (that is, $\nu(F) \geq \epsilon\mu(F)$ for all $F \subseteq E$; or $\nu - \epsilon\mu$ is a signed measure, and E is a positive set.

Proof: Consider a signed measure $\nu - 1/n \cdot \mu$, and then look at a Hahn decomposition for $\nu - 1/n \cdot \mu$, $X = P_n \dot{\cup} N_n$. Then, P_n is a positive set for $\nu - 1/n \cdot \mu$. Let $P = \bigcup_n^\infty P_n$ and $N = \bigcap_n^\infty N_n$ $= \bigcap_n^\infty P_n^c = P^c$. So, $X = P \dot{\cup} N$. Since $N \subsetneq N_n$ for all n, $(\nu - 1/n \cdot \mu)(N) \leq 0 \Rightarrow \nu(N) \leq 1/n \cdot \mu(N)$ for all $n \Rightarrow$ $\nu(N) = 0$. So, if $\mu(P) = 0$, then $\mu \perp \nu$. On the other hand, if $\mu(P) > 0$, then there exist n_0 such that $\mu(P_{n_0}) > 0$. But, P_{n_0} is positive for $\nu - 1/n_0 \cdot \mu$. Take $\epsilon = 1/n_0$ and $E = P_{n_0}$, then we are done.

Definition: Let ν be a signed measure. Then, ν is **σ-finite** $\Leftrightarrow$ $|\nu|$ is σ-finite.

Theorem 3.8 (The Lebesgue-Radon-Nikodym Theorem): Let $(X, \mathcal{M})$ be a measurable space with μ which is positive and σ-finite, and ν which is also positive and σ-finite. Then, there exist λ and ρ positive such that $\nu = \lambda + \rho$, $\lambda \perp \mu$ and $\rho \ll \mu$. (Lebesgue decomposition)
Moreover, there exists $f \geq 0$ which is $\mathcal{M}$-measurable such that $\rho(E) = \int_E f d\mu$ for all $E \in \mathcal{M}$.

Proof: <u>Case 1</u>: Assume that μ and ν are both finite.
Let $\mathcal{F} = \{f : X \to [0, +\infty] : f$ is $\mathcal{M}$-measurable and $\int_E f d\mu \leq \nu(E)$ for all $E \in \mathcal{M}\}$ Note that zero function is in $\mathcal{F}$, so $\mathcal{F} \neq \emptyset$. Next, let $a = sup\{\int_X f d\mu : f \in \mathcal{F}\}$, and pick $f_n \in \mathcal{F}$ such that $\int_X f_n d\mu \to a$. Let $f = \sup_n f_n$, then f is $\mathcal{M}$-measurable by Proposition 2.7. Let $f, h \in \mathcal{F}$, $g = max\{f, h\}$, and $A = \{x : g(x) = f(x)\}$. Then, on A^c, $g(x) = h(x)$. So, $\int_E g d\mu = \int_{E\cap A} g d\mu + \int_{E \cap A^c} g d\mu = \int_{E\cap A} f d\mu + \int_{E\cap A^c} h d\mu \leq$ $\nu(E \cap A) + \nu(E \cap A^c) = \nu(E)$. Thus, $g = max\{f, h\} \in \mathcal{F}$. Inductively, let $g_n = max\{f_1, f_2, \cdots, f_n\} \in \mathcal{F}$, and $g_1 \leq$ $g_2 \leq \cdots \leq f$. So, $g_n \nearrow f$. So, if $E \in \mathcal{M}$, then $\int_E f d\mu$ $= \lim_n \int_E g_n d\mu \leq \nu(E)$, and so $f = \sup_n\{f_n\} \in \mathcal{F}$. Now, a $\geq$ $\int_X f d\mu = \lim_n \int_X g_n d\mu \geq \lim_n \int_X f_n d\mu = a$. Thus, $\int_X f d\mu = a$, with ~~and~~ so $f \in \mathcal{F}$. Let $\rho(E) = \int_E f d\mu$, then $\rho \ll \mu$.

Next, let $\lambda = \nu - \rho$. [Show that $\lambda \perp \mu$.]
Suppose that λ and μ are not mutually singular. Then, by Theorem 3.7, there exists $\epsilon > 0$, $E \in \mathcal{M}$ with $\mu(E) > 0$ such that $(\lambda - \epsilon\mu)(F) \geq 0$ for all $F \subseteq E \Rightarrow$ for all $A \in \mathcal{M}$, $\lambda(A) \leq \lambda(A \cap E) \geq \epsilon\mu(A \cap E)$. Also, $\lambda(A) - \nu(A) - \rho(A) - \nu(A) - \int_A f d\mu \geq \epsilon \int_A \chi_E d\mu \Rightarrow \nu(A) \geq \int_A (f + \epsilon\chi_E) d\mu$ for all $A \in \mathcal{M} \Rightarrow f + \epsilon\chi_E \in \mathcal{F} \Rightarrow a \geq \int_X (f + \epsilon\chi_E) d\mu = a + \epsilon\mu(E) > a$. But, this is a contradiction. Thus, $\lambda \perp \mu$.
Case 2: Assume that μ and ν are both σ-finite. Then, if $X = \dot{\bigcup_i} A_i$, then $\mu(A_i) < +\infty$, and if $X = \dot{\bigcup_j} B_j$, then $\nu(B_j) < +\infty$
$\Rightarrow \dot{\bigcup_{i,j}} (A_i \cap B_j)$, $\mu(A_i \cap B_j) < +\infty$ and $\nu(A_i \cap B_j) < +\infty$.
Now, apply Case 1 to get, f_{ij} on $A_i \cap B_j$ such that for all $E \subseteq A_i \cap B_j$, $\nu(E) = \int_E f_{i,j} d\mu + \lambda_{ij}(E)$ where $\lambda_{ij} \perp \mu$ on $A_i \cap B_j$ (That is, there exists $W_{i,j} \dot{\cup} Z_{i,j}$ such that $\lambda_{i,j}(W_{i,j}) = 0$ and $\mu(Z_{i,j}) = 0$.) Define $f : X \to [0, +\infty]$ by $f(x) = f_{i,j}(x)$ when $x \in A_i \cap B_j$. Next, set $\lambda(E) = \sum_{i,j} \lambda_{i,j}(E \cap (A_i \cap B_j)) \geq 0$, $\lambda(\cup W_{i,j}) = 0$, $\mu(\cup Z_{i,j}) = 0$, and $X = (\cup W_{i,j}) \dot{\cup} (\cup Z_{i,j})$. Then, $\lambda \perp \mu$. Define $\rho(E) = \int_E f d\mu = \sum_{i,j} \int_{E \cap (A_i \cap B_j)} f_{i,j} d\mu$,

then $\nu = \rho + \lambda$.

Note that some results hold when ν is a signed measure. Apply the above argument to $|\nu| = \nu^+ + \nu^-$.

Corollary (Radon-Nikodym): Let ν and μ be both positive and σ-finite measures with $\nu \ll \mu$, then there exists $f : X \to [0, +\infty]$ which is $\mathcal{M}$-measurable such that $\nu(E) = \int_E f d\mu$.

Proof: By Theorem 3.8, there exist λ and ρ both positive such that $\nu = \lambda + \rho$ where $\lambda \perp \mu$ and $\rho(E) = \int_E f d\mu$. Since $\lambda \perp \mu$, $X = A \dot{\cup} B$ with $\lambda(A) = 0$ and $\mu(B) = 0$. Given any $E \in \mathcal{M}$, $0 \leq \lambda(E) = \lambda(E \cap B) + \lambda(E \cap A) = \lambda(E \cap B) \leq \nu(E \cap B)$ But, $\mu(E \cap B) = 0$ since $E \cap B \subseteq B$. So, $\nu \ll \mu \Rightarrow \nu(E \cap B) = 0 \Rightarrow \lambda(E) = 0$ for all $E \in \mathcal{M} \Rightarrow \lambda = 0 \Rightarrow \nu(E) = \rho(E) = \int_E f d\mu$.