Chapter 3 Signed Measures and Differentiation Theory

We will study signed measures, differentiation of measures and differentiation theory of functions in this chapter.

3.1 Signed Measures

Definition: Let $(X, \mathcal{M})$ be a measurable space, then $\nu : \mathcal{M} \to \overline{\mathbb{R}}$ is a **signed measure** if the following conditions are satisfied:

(1) $\nu(\emptyset) = 0$

(2) ν can assume positive and negative values, but only one of $+\infty$ and $-\infty$.

(3) If $\{E_j\}$ is a collection of disjoint measurable sets, then $\nu(\bigcup_{j=1}^{\infty} E_j) = \sum_{j=1}^{\infty} \nu(E_j)$ where whenever $\nu(\bigcup_{j=1}^{\infty} E_j)$ is finite, then the series converges absolutely, and when $\nu(\bigcup_{j=1}^{\infty} E_j)$ is infinite, then the series properly diverges to $\nu(\bigcup_{j=1}^{\infty} E_j) = \pm\infty$. That is, $\nu(\bigcup_{j=1}^{\infty} E_j) = +\infty \Rightarrow$ for all $C > 0$, there exists N such that $\sum_{j=1}^{\infty} \nu(E_j) > C$ for all $n \geq N$.

finite signed measure if a signed measure $\nu(E)$ never $\pm\infty$

Example (1): Let $(X, \mathcal{M}, \mu_1)$ and $(X, \mathcal{M}, \mu_2)$ be measure spaces, and at least one them is finite, then $\nu(E) = \mu_1(E) - \mu_2(E)$ is a signed measure, and $\sum_{j=1}^{\infty} [\mu_1(E_j) - \mu_2(E_j)] = \sum_{j=1}^{\infty} \mu_1(E_j) - \sum_{j=1}^{\infty} \mu_2(E_j)$. both finite then a finite signed measure

Example (2): Let $(X, \mathcal{M}, \mu)$ be a measure space and $f : X \to \overline{\mathbb{R}}$ be measurable. Look at f^+ and f^- and assume that one of the integrals $\int_X f^+ d\mu$ and $\int_X f^- d\mu$ is finite. Then, setting $\nu(E) = \int_E f d\mu$ is a signed measure. If $\nu^+(E) = \int_E f^+ d\mu$ and $\nu^-(E) = \int_E f^- d\mu$, then ν^+ and ν^- are measures, and $\nu(E) = \nu^+(E) - \nu^-(E)$. Split X into two pieces with positive measures.

Both integrals finite, i.e., $f \in \mathcal{L}^1$ then a finite signed measure

Proposition 3.1: Suppose that ν is a signed measure and $E_1 \subseteq E_2 \subseteq \cdots$, then $\nu(\bigcup_{j=1}^{\infty} E_j) = \lim_j \nu(E_j)$. Also, if $E_1 \supseteq E_2 \supseteq \cdots$ with $\nu(E_1) \neq \pm\infty$, then $\nu(\bigcap_{j=1}^{\infty} E_j) = \lim_j \nu(E_j)$.

Proof: Suppose that ν is a signed measure. Let $E_0 = \emptyset$, and $E = E_1 \dot\cup (E_2 \backslash E_1) \dot\cup (E_3 \backslash E_2) \dot\cup \cdots$. Then, $E = \dot{\bigcup}_{j=0}^{\infty} (E_{j+1} \backslash E_j)$ $= \dot{\bigcup}_{j=1}^{\infty} E_j$, and $\nu(E) = \nu(\dot{\bigcup}_{j=1}^{\infty} E_j) = \nu(\dot{\bigcup}_{j=0}^{\infty} (E_{j+1} \backslash E_j)) =$ $\sum_{j=0}^{\infty} \nu(E_{j+1} \backslash E_j)$. Assume that $\nu(E) \neq \pm\infty$, is finite.

Then, $\sum_{j=0}^{\infty} \nu(E_{j+1} \backslash E_j) = \nu(E)$ converges absolutely $\Rightarrow$ $\sum_{j=0}^{\infty} |\nu(E_{j+1} \backslash E_j)| \leq +\infty$. So, given $\epsilon > 0$, pick N such that $|\nu(E) - \sum_{j=0}^{n} \nu(E_{j+1} \backslash E_j)| < \epsilon$ for all $n \geq N$. Suppose that $\sum_{j=0}^{n} \nu(E_{j+1} \backslash E_j) = \nu(E_n)$, then $|\nu(E) - \nu(E_n)| < \epsilon$ for all $n \geq N \Rightarrow \nu(E) = \nu(\bigcup_{n=1}^{\infty} E_n) = \lim_n \nu(E_n)$.

Next, suppose that $\nu(E)$ is infinite.

<u>Case $\nu(E) = +\infty$</u>: $\nu(E) = \sum_{j=0}^{\infty} \nu(E_{j+1} \backslash E_j)$ means that the series properly diverges to $+\infty$. That is, given $C > 0$, there exists N such that $\nu(E_n) = \sum_{j=0}^{n} \nu(E_{j+1} \backslash E_j) > C$ for all $n > N \Rightarrow \lim_n \nu(E_n) = +\infty = \nu(E)$.

<u>Case $\nu(E) = -\infty$</u>: Similarly, $\lim_n \nu(E_n) = -\infty = \nu(E)$.

Next, suppose that $E_1 \supseteq E_2 \supseteq \cdots$, and let $E = \bigcap_{j=1}^{\infty} E_j$. Then, $E_1 = E \,\dot\cup\, (E_1 \backslash E_2) \,\dot\cup\, (E_2 \backslash E_3) \,\dot\cup \cdots \Rightarrow \nu(E_1) = \nu(E) + \sum_{j=0}^{\infty} \nu(E_j \backslash E_{j+1}) \Rightarrow \nu(E_1) - \nu(E) = \sum_{j=0}^{\infty} \nu(E_j \backslash E_{j+1})$. If $\nu(E)$ is finite, $\nu(E_1) - \nu(E) = \sum_{j=0}^{\infty} \nu(E_j \backslash E_{j+1}) \Rightarrow |\, \nu(E_1) - \nu(E) - \sum_{j=0}^{n} \nu(E_j \backslash E_{j+1})| < \epsilon$ for all $n \geq N$. Now, $\sum_{j=0}^{n} \nu(E_j \backslash E_{j+1}) = \nu(E_1) - \nu(E_n) \Rightarrow |\nu(E) - \nu(E_n)| < \epsilon$ for all $n \geq N$. Thus, $\nu(E) = \nu(\bigcap_{n=1}^{\infty} E_n) = \lim_n \nu(E_n)$.

Do cases for when $\nu(E)$ is infinite.

Definition: Let $(X, \mathcal{M})$ be a measurable space, and ν be a signed measure. Then, $E \in \mathcal{M}$ is called **ν-positive** (or positive for ν) provided that $F \subseteq E$ and $F \in \mathcal{M} \Rightarrow \nu(F) \geq 0$. Similarly, E is called **ν-negative** provided that $F \subseteq E$ and $F \in \mathcal{M} \Rightarrow \nu(F) \leq 0$. Also, E is **ν-null** provided that $F \subseteq E$ and $F \in \mathcal{M} \Rightarrow \nu(F) = 0$.

Example: Let $f \in \mathcal{L}^1(\mu)$, and write $f = f^+ - f^-$. Define $\nu(E) = \int_E f d\mu$. Then, ν is a signed measure. Suppose that $A^+ = \{x : f(x) > 0\}$, $A^- = \{x : f(x) < 0\}$ and $A^0 = \{x : f(x) = 0\}$. Then, A^+ is ν-positive, A^- is ν-negative, and A^0 is ν-null.

Lemma 3.2: Let ν be a signed measure. Then:

(1) If P is ν-positive, $Q \subseteq P$ and $Q \in \mathcal{M}$, then Q is ν-positive.

(2) If each P_n is ν-positive, then $\bigcup_{n=1}^{\infty} P_n$ is ν-positive.

Note that (1) and (2) are also true for ν-negative and ν-null.

Theorem 3.3 (The Hahn Decomposition Theorem): Let ν be a signed measure on $(X, \mathcal{M})$. Then, there exist P which is ν-positive and N which is ν-negative, and $P, N \in \mathcal{M}$ with $X = P \,\dot\cup\, N$. If there are P' and N' such that $X = P' \,\dot\cup\, N'$, then $P \triangle P'$ and $N \triangle N'$ are both ν-null.

Proof: Without loss of generality, assume that $+\infty$ is omitted. Let $m = sup\{\nu(E) : E \text{ is } \nu\text{-positive}\}$, also let P_n be ν-positive such that $\nu(P_n) \to m$, and $P = \bigcup_{n=1}^{\infty} P_n$. Then, P is ν-positive.

This implies that $\nu(P) \leq m$. However, $P_n \subseteq P$, an so $P = P_n \dot{\cup} (P \backslash P_n) \Rightarrow \nu(P) = \nu(P_n) + \nu(P \backslash P_n) \geq \nu(P_n) \Rightarrow \nu(P) = m \Rightarrow m < +\infty$.

Next, let $N = X \backslash P$. [Show that N is ν-negative.]

Suppose that $A \subseteq N$ and A is ν-positive $\Rightarrow A \cup P$ is ν-positive $\Rightarrow m \geq \nu(A \cup P) = \nu(A) + \nu(P) = \nu(A) + m \Rightarrow \nu(A) = 0$. Suppose that $B \subseteq A$, then $A = B \dot{\cup} (A \backslash B) \Rightarrow 0 = \nu(A) = \nu(B) + \nu(A \backslash B) \Rightarrow \nu(B) = 0 \Rightarrow A$ is ν-null.

Suppose that N is not ν-negative $\Rightarrow$ There exists $A \subseteq N$ such that $\nu(A) > 0 \Rightarrow A$ is not ν-positive (otherwise $\nu(A) = 0$) $\Rightarrow$ There exists $B \subseteq A$ with $\nu(B) < 0$. Let $A_1 = A \backslash B \Rightarrow A = B \dot{\cup} A_1 \Rightarrow \nu(A) = \nu(B) + \nu(A_1) < \nu(A_1)$ and A_1 is not ν-positive. [Now, set up an inductive process.]

Inductively, let n_1 = the least integer such that there exists $B \subseteq N$ with $\nu(B) > 1/n_1$. Pick such a set, and call it A_1 so that $A_1 \subseteq N$ and $\nu(A_1) > 1/n_1$. By the above argument, we saw that A_1 cannot be ν-positive, and that there exists a set $B \subseteq A_1$ with $\nu(B) \geq \nu(A_1)$. Let n_2 = the least integer such that there exists $B \subseteq A_1$ and $\nu(B) > \nu(A_1) + 1/n_2$. Again pick a such a set, and call it A_2 so that $A_2 \subseteq A_1$ and $\nu(A_2) > \nu(A_1) + 1/n_2$. Inductively, n_j = the least integer such that there exists $B \subseteq A_{j-1}$ with $\nu(B) > 1/n_j$, $A_j \subseteq A_{j-1}$, and $\nu(A_j) > \nu(A_{j-1}) + 1/n_j$. Now, let $A = \bigcap_{j=1}^{\infty} A_j$. Since $\nu(A_1)$ is finite, $\nu(A) = \lim_j \nu(A_j)$ and $0 \leq \nu(A) \Rightarrow \nu(A) < +\infty$ (by the assumption.) Now, $\nu(A_j) > \nu(A_{j-1}) + 1/n_j > \nu(A_{j-2}) + 1/n_{j-1} + 1/n_j \Rightarrow \nu(A_j) \geq \sum_{k=1}^{j} 1/n_k \Rightarrow \sum_{k=1}^{\infty} 1/n_k \leq \nu(A) < +\infty \Rightarrow n_k \to +\infty$ as $k \to +\infty$. Since $A \subseteq N$, we know that A is not ν-positive $\Rightarrow$ There exists $B \subseteq A$ such that $\nu(B) > \nu(A) \Rightarrow$ There exists n such that $\nu(B) > \nu(A) + 1/n \Rightarrow$ There exists k such that $n_k > n$. So, we have $\nu(B) > \nu(A) + 1/n > \nu(A_{k-1}) + 1/n > \nu(A_{k-1}) + 1/n_k$. So, $B \subseteq A \subseteq A_{k-1}$. But, this is a contradiction. Thus, N is ν-negative, and so we have shown that $X = P \dot{\cup} (X \backslash P) = P \dot{\cup} N$ where P is ν-positive and N is ν-negative.

Suppose that $X = P' \dot{\cup} N'$ where P' is ν-positive and N' is ν-negative. Then, $P \backslash P' \subseteq P \Rightarrow P \backslash P'$ is ν-positive and $P \backslash P' \subseteq N' \Rightarrow P \backslash P'$ is ν-negative. Thus, $P \backslash P'$ is ν-null because every subsets of $P \backslash P'$ must have measure 0 to satisfy above conditions. Similarly, $P' \backslash P \subseteq P'$ and $P' \backslash P \subseteq N$, and so $P' \backslash P$ is also ν-null. Thus, $P \triangle P' = (P \backslash P') \cup (P' \backslash P)$ is ν-null. Similarly, $N \triangle N'$ is ν-null.

Definition: Given a signed measure ν, write $X = P \dot{\cup} N$ where P is ν-positive and N is ν-negative. Then, this decomposition is called a **Hahn decomposition for ν.**

Note that given a Hahn decomposition for ν, if we set $\nu_1(E) = \nu(E \cap P)$ and $\nu_2(E) = -\nu(E \cap N)$. Then, ν_1 and ν_2 are positive measures and $\nu = \nu_1 - \nu_2$. This gives a way to express ν as a difference of positive measures.

Definition (definition of orthogonality for measure): Let μ and ν be signed measures. We say that they are **mutually singular**, denoted by $\mu \perp \nu$, if $X \doteq E \dot{\cup} F$ such that E is μ-null (that is, μ lives in F) and F is ν-null (that is, ν lives in E.)

Theorem 3.4 (The Jordan Decomposition Theorem): Let ν be a signed measure. Then, there exist unique positive measure ν^+ and ν^- such that $\nu = \nu^+ - \nu^-$ and $\nu^+ \perp \nu^-$.

Proof: Take any Hahn decomposition $X = P \dot{\cup} N$, and define $\nu_1(E) = \nu(E \cap P)$ and $\nu_2(E) = -\nu(E \cap N)$. Then, ν_1 and ν_2 are positive measures and $\nu = \nu_1 - \nu_2$. Since P is ν_2-null and N is ν_1-null, $\nu_1 \perp \nu_2$.
[Show uniqueness.]
Suppose that $\nu = \mu_1 - \mu_2$ where μ_1 and μ_2 are both positive, and $\mu_1 \perp \mu_2 \Rightarrow X = E \dot{\cup} F$ where E is μ_2-null and F is μ_1-null. So, if $A \subseteq E$, then $\nu(A) = \mu_1(A) - \mu_2(A) = \mu_1(A) \geq 0$ $\Rightarrow E$ is ν-positive. Similarly, if $A \subseteq F$, then $\nu(A) = \mu_1(A) - \mu_2(A) = -\mu_2(A) \leq 0 \Rightarrow F$ is ν-negative $\Rightarrow X = E \dot{\cup} F$ is a Hahn decomposition $\Rightarrow P \triangle E$ and $N \triangle F$ are both ν-null sets. If $A \in \mathcal{M}$, then $\mu_1(A) = \mu_1(A \cap E) + \mu_1(A \cap F) = \mu_1(A \cap E) = \nu(A \cap E) = \nu(A \cap P) = \nu_1(A)$. Thus, $\mu_1 = \nu_1$. Similarly, $\mu_2 = \nu_2$, and so ν is unique.

<u>Note</u>: This proof shows that taking any Hahn decomposition $X = P \dot{\cup} N$, setting $\nu^+(E) = \nu(E \cap P)$ and $\nu^-(E) = -\nu(E \cap N)$ gives $\nu = \nu^+ - \nu^-$, $\nu^+ \perp \nu^-$ and it is unique.

Definition: Let $X = \ = P \dot{\cup} N$ be a Hahn decomposition. Set $\nu^+(E) = \nu(E \cap P)$ and $\nu^-(E) = -\nu(E \cap N)$. Then, $\nu = \nu^+ - \nu^-$ is called the **Jordan decomposition** of ν, and ν^+ and ν^- are called the **positive** and **negative variations** of ν. The measure, $|\nu| = \nu^+ + \nu^-$ is called the **total variation** of ν.

Example: Let μ be a positive measure, $f \in \mathcal{L}^1(\mu)$, and set $\nu(E) = \int_E f d\mu$. Then, $\nu^+(E) = \int_E f^+ d\mu$ and $\nu^-(E) = \int_E f^- d\mu$, and $|\nu|(E) = \int_E |f| d\mu$. Note that $|\nu(E)| = |\nu^+(E) - \nu^-(E)| \leq \nu^+(E) + \nu^-(E) = |\nu|(E)$. In general, they are not equal.