Theorem 2.37 (Fubini-Tonelli): Let $(X, \mathcal{M}, \mu)$ and $(Y, \mathcal{N}, \nu)$ be both σ-finite.

(a) (Tonelli): Let $f : X \times Y \to \overline{\mathbb{R}}$, $f \geq 0$ and $\mathcal{M} \otimes \mathcal{N}$-measurable, then $g(x) = \int_Y f_x d\nu(y)$ is $\mathcal{M}$-measurable and $h(y) = \int_X f^y d\mu(x)$ is $\mathcal{N}$-measurable, and $\int_X [\int_Y f_x d\nu(y)] d\mu(x) = \int_Y [\int_X f^y d\mu(x)] d\nu(y)$.

(b) (Fubini): Let $f : X \times Y \to \mathbb{R}$ be $\mathcal{M} \otimes \mathcal{N}$-measurable. Suppose that $f \in \mathcal{L}^1(\mu \times \nu)$, then $f_x \in \mathcal{L}^1(\nu)$ *a.e.* x and $f^y \in \mathcal{L}^1(\mu)$ *a.e.* y, and $\int_X [\int_Y f_x d\nu(y)] d\mu(x) = \int_Y [\int_X f^y d\mu(x)] d\nu(y)$.

Note: The following example show that why we need "*a.e.* x"

Let $X = Y = \mathbb{R}$, and $\mu = \nu =$ Lebesgue measure. Suppose that $f = +\chi_{\{x_0\} \times \mathbb{R}} - \chi_{\mathbb{R} \times \{y_0\}}$. Then, $f \in \mathcal{L}^1(\mu \times \nu)$ because f is 0 except on $x = x_0$ or $y = y_0$. As $x \neq x_0 \Rightarrow f_x(y) = \begin{cases} 0 & \text{if } y \neq y_0 \\ -1 & \text{if } y = y_0 \end{cases}$

But, $f_{x_0}(y) = \begin{cases} 1 & \text{if } y \neq y_0 \\ 0 & \text{if } y = y_0 \end{cases}$ So, $f_{x_0}(y) \notin \mathcal{L}^1(\nu)$

Proof of (a) (Tonelli): <u>Case 1</u> Suppose that $f = \chi_E$ and $E \in \mathcal{M} \otimes \mathcal{N}$. Then, $f_x = (\chi_E)_x = \chi_{E_x}$ and $g(x) = \int_Y \chi_{E_x} d\nu = \nu(E_x)$ which is measurable by Theorem 2.36. Also, $h(y) = \int_X \chi_{E_y} d\mu = \mu(E^y)$ which is also measurable by Theorem 2.36. Moreover, $\int_X g(x) d\mu(x) = \int_X \nu(E_x) d\mu(x) = \int_Y \mu(E^y) d\nu(y) = \int_Y h(y) d\mu(y)$. Thus, (a) is true in this case.

<u>Case 2</u>: Now, let $f = \sum_{j=1}^{n} a_j \chi_{E_j}$ where $a_j \geq 0$, and $E_j \in \mathcal{M} \otimes \mathcal{N}$.

Then, $g(x) = \sum_{j=1}^{n} a_j \nu((E_j)_x)$ and $h(y) = \sum_{j=1}^{n} a_j \mu((E_j)^y)$ are both measurable. Also, $\int_X g(x) d\mu(x) = \sum_{j=1}^{n} a_j \int_X \nu((E_j)_x) d\mu(x) = \sum_{j=1}^{n} \int_Y \mu((E_j)^y) d\nu(y) = \int_Y h(y) d\mu(y)$. So, (a) is true when f is simple.

<u>Case 3</u>: Let $f \geq 0$ and $\mathcal{M} \otimes \mathcal{N}$-measurable. Then, there exists $\phi_1 \leq \phi_2 \leq \cdots \leq f$ where each ϕ_n is simple and $\lim_n \phi_n(x, y) = f(x, y) \Rightarrow g(x) = \int_Y f_x(y) d\nu(y) \overset{*}{=} \lim_n \int_Y (\phi_n)_x(y) d\nu(y)$ and $h(y) = \int_X f^y(x) d\mu(x) \overset{*}{=} \lim_n \int_X (\phi_n)^y(x) d\mu(x)$.

Thus, $\int_X g(x) d\mu(x) = \int_X [\lim_n \int_Y (\phi_n)_x(y) d\nu(y)] d\mu(x) \overset{*}{=}$

$\lim_n \int_X \int_Y (\phi_n)_x(y) d\nu(y) d\mu(x) \overset{**}{=} \lim_n \int_Y \int_X (\phi_n)^y(x) d\mu(x) d\nu(y)$
$\overset{*}{=} \int_Y [\lim_n \int_X (\phi_n)^y(x) d\mu(x)] d\nu(y) = \int_Y h(y) d\nu(y)$, and so
Tonelli is true.
[Note that $\overset{*}{=}$ is true by the Monotone Convergence Theorem,
and $\overset{**}{=}$ is true by the Case 2.]

Proof of (b) (Fubini): By Tonelli, $h \geq 0 \Rightarrow$ If $g(x) = \int h(x,y) d\nu(y)$,
then $\int_X g(x) d\mu(x) = \int_{X\times Y} h d(\mu \times \nu)$. So, if $\int_{X\times Y} h d(\mu\times\nu)$
$< +\infty$, then $\int_X g(x) d\mu(x) < +\infty \Rightarrow \{x : g(x) = +\infty\}$ has
measure 0. So, $f \in \mathcal{L}^1(\mu \times \nu) \to \int_{X\times Y} |f| d(\mu\times\nu) < +\infty$
$\Rightarrow \int_{X\times Y} f^+ d(\mu\times\nu) < +\infty$ and $\int_{X\times Y} f^- d(\mu\times\nu) < +\infty$.
Let $g^+(x) = \int_Y f^+(x,y) d\nu(y)$ and $g^-(x) = \int_Y f^-(x,y) d\nu(y)$
$\Rightarrow$ If $N^+ = \{x : g^+(x) = +\infty\}$ and $N^- = \{x : g^-(x) =$
$+\infty\}$, then $\mu(N^+) = \mu(N^-) = 0 \Rightarrow$ If $N = N^+ \cup N^-$, then
$\mu(N) = 0$. For $x \notin N$, $g^+(x)$ and $g^-(x)$ are both finite. So,
$g^+(x) - g^-(x) = \int_Y f^+(x,y) d\nu(y) - \int_Y f^-(x,y) d\nu(y) =$
$\int_Y f(x,y) d\nu(y) = g(x)$ is defined for all $x \notin N$.
Finally, $\int_{X\times Y} f^+ d(\mu\times\nu) = \int_X [\int_Y f^+(x,y) d\nu(y)] d\mu(x) =$
$\int_{X\setminus N} [\int_Y f^+(x,y) d\nu(y)] d\mu(x)$. Similarly, $\int_{X\times Y} f^- d(\mu\times\nu)$
$= \int_{X\setminus N} [\int_Y f^-(x,y) d\nu(y)] d\mu(x) \Rightarrow \int_{X\times Y} f d(\mu\times\nu) =$
$\int_{X\times Y} f^+ d(\mu\times\nu) - \int_{X\times Y} f^- d(\mu\times\nu) =$
$\int_{X\setminus N} [\int_Y f(x,y) d\nu(y)] d\mu(x) = \int_X [\int_Y f(x,y) d\nu(y)] d\mu(x)$.
Similarly, iteration in other direction is also true.

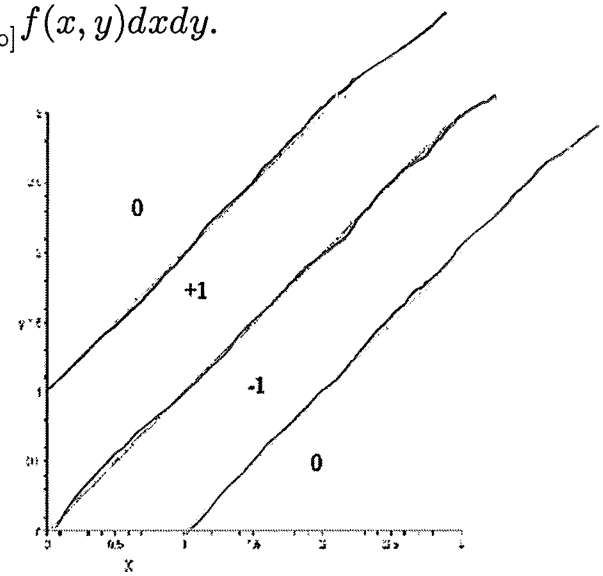
Caution: If $f \in \mathcal{L}^1(\mu\times\nu)$, then by Fubini $\int_X [\int_Y f(x,y) d\nu(y)] d\mu(x) =$ $\int_Y [\int_X f(x,y) d\mu(x)] d\nu(y)$. Note that if $\int_X [\int_Y f(x,y) d\nu(y)] d\mu(x)$ and $\int_Y [\int_X f(x,y) d\mu(x)] d\nu(y)$ are both finite, but $\int_X [\int_Y f(x,y) d\nu(y)] d\mu(x) \neq$ $\int_Y [\int_X f(x,y) d\mu(x)] d\nu(y) \Rightarrow f \notin \mathcal{L}^1(\mu\times\nu)$. The following example illustrate the problem:

Let $X = Y = \mathbb{R}^+$ and $\mu = \nu = m$, Lebesgue measure.
Consider the graph in the next page.
It is clear that $f \notin \mathcal{L}^1(m \times m)$.
First, $g(x) = \int_{[0,+\infty]} f(x,y) dy = \begin{cases} 0 & \text{if } x \geq 1 \\ \text{something} & \text{if } x < 1 \end{cases}$ and
$\int_{[0,+\infty]} \int_{[0,+\infty]} f(x,y) dy dx = \int_{[0,1]} [\int_{[0,+\infty]} f(x,y) dy] dx = 1/2$.

On the other hand,

$$h(y) = \int_{[0,+\infty]} f(x,y)dx = \begin{cases} 0 & \text{if } y \geq 1 \\ \text{something} & \text{if } y < 1 \end{cases} \text{ and}$$

$\int_{[0,+\infty]}\int_{[0,+\infty]} f(x,y)dxdy = \int_{[0,1]}[\int_{[0,+\infty]} f(x,y)dx]dy - -1/2.$
Thus, $\int_{[0,+\infty]}\int_{[0,+\infty]} f(x,y)dydx$ and $\int_{[0,+\infty]}\int_{[0,+\infty]} f(x,y)dxdy$ are both finite, but $\int_{[0,+\infty]}\int_{[0,+\infty]} f(x,y)dydx = 1/2 \neq -1/2 = \int_{[0,+\infty]}\int_{[0,+\infty]} f(x,y)dxdy.$



Let $(X, \mathcal{M}, \mu)$ and $(Y, \mathcal{N}, \nu)$ be σ-finite measure spaces, and get $\mu \times \nu$ on $X \times Y$ with $\mathcal{M} \otimes \mathcal{N}$. Complete this measure (that is, throw in the sets of measure 0) to get $\widetilde{\mu \times \nu} = \lambda$ and a bigger σ-algebra $\widetilde{\mathcal{M} \otimes \mathcal{N}} = \mathcal{L}$.
Now, we can state the Theorem 2.37 in more general.

Theorem 2.39 (Fubini-Tonelli): Let $f : X \times y \to \overline{\mathbb{R}}$ be $\mathcal{L}$-measurable.
Then, f_x is $\mathcal{N}$-measurable $a.e.\ x$, and f^y is $\mathcal{M}$-measurable $a.e.\ y$.
The rest is the same as Theorem 2.37.