Theorem 2.36: Let $(X, \mathcal{M}, \mu)$ and $(Y, \mathcal{N}, \nu)$ be σ-finite measure spaces. If $E \in \mathcal{M} \otimes \mathcal{N}$, then:

(1) $x \to \nu(E_x)$ is $\mathcal{M}$-measurable, and $y \to \mu(E^y)$ is $\mathcal{N}$-measurable.

(2) $\int_X \nu(E_x) d\mu(x) = \int_Y \mu(E^y) d\nu(y)$

(3) If we set $\mu \times \nu(E) = \int_X \nu(E_x) d\mu(x)$, then $\mu \times \nu$ is a measure on $\mathcal{M} \otimes \mathcal{N}$.

Proof: First assume that $\nu(Y) < +\infty$ and $\mu(X) < +\infty$, and let $\mathcal{C} = \{E \in \mathcal{M} \otimes \mathcal{N} : \text{(1) and (2) hold.}\}$

<u>Claim</u>: If $E = A \times B$, $A \in \mathcal{M}$ and $B \in \mathcal{N}$, then $E \in \mathcal{C}$.

Proof of Claim: Let $E = A \times B$, $A \in \mathcal{M}$ and $B \in \mathcal{N}$. Then, $E_x = (A \times B)_x = \begin{cases} B & \text{if } x \in A \\ \emptyset & \text{if } x \notin A \end{cases}$ and so $x \to \nu(E_x)$ $= \begin{cases} \nu(B) & \text{if } x \in A \\ 0 & \text{if } x \notin A \end{cases} = \nu(B)\chi_A(x)$. Also, $y \to \mu(E^y)$ $= \mu(A)\chi_B(y)$. Thus, (1) hold for $E = A \times B$, $A \in \mathcal{M}$ and $B \in \mathcal{N}$. Now, if $E = A \times B$, then $\int_X \nu(E_x) d\mu(x)$ $= \int_X \nu(B)\chi_A d\mu(x) = \nu(B)\mu(A)$ and $\int_Y \mu(E^y) d\nu(y)$ $= \int_Y \mu(A)\chi_B d\nu = (y)\mu(A)\nu(B)$. Thus, (2) also hold for $E = A \times B$, and so $E = A \times B \in \mathcal{C}$.

Next, suppose that $E =$ a finite disjoint union of rectangles $= \dot{\bigcup}_{j=1}^{n} (A_j \times B_j)$. Then, $E_x = \dot{\bigcup}_{j=1}^{n} (A_j \times B_j)_x$ is still a finite disjoint union $\Rightarrow \nu(E_x) = \nu(\dot{\bigcup}_{j=1}^{n} (A_j \times B_j)_x) = \sum_{j=1}^{n} \nu(A_j \times B_j)_x$ is measurable. Thus, (1) holds for $E = \dot{\bigcup}_{j=1}^{n} (A_j \times B_j)$. Also, when $E = \dot{\bigcup}_{j=1}^{n} (A_j \times B_j)$, $\nu(E_x) = \sum_{j=1}^{n} \nu(A_j \times B_j)_x = \sum_{j=1}^{n} \nu(B_j)\chi_{A_j}(x)$ $\Rightarrow \int_X \nu(E_x) d\mu(x) = \int_X \sum_{j=1}^{n} \nu(B_j)\chi_{A_j} d\mu(x,) = \sum_{j=1}^{n} \mu(A_j)\nu(B_j)$.

Similarly, if $E = \dot{\bigcup}_{j=1}^{n} (A_j \times B_j)$, then $\mu(E^y) = \sum_{j=1}^{n} \mu(A_j \times B_j)^y$ $= \sum_{j=1}^{n} \mu(A_j)\chi_{B_j}(y) \Rightarrow \int_Y \mu(E^y) d\nu(y) = \int_Y \sum_{j=1}^{n} \mu(A_j)\chi_{B_j} d\nu(y)$ $= \sum_{j=1}^{n} \mu(A_j)\nu(B_j)$. Thus, (2) holds, and so $E = \dot{\bigcup}_{j=1}^{n} (A_j \times B_j)$ $\in \mathcal{C}$.

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Claim: Let $\mathcal{A} = \{\dot{\bigcup}_{j=1}^{n} (A_j \times B_j) : A_j \in \mathcal{M}$ and $B_j \in \mathcal{N}\}$ Then, $\mathcal{A}$ is an algebra.

Proof of Claim: Consider $(A \times B) \cup (C \times D) = (A \cap C) \times (B \cap D) \dot{\cup} (A \backslash C) \times B \dot{\cup} (C \backslash A) \times D \dot{\cup} (A \cap C) \times (B \backslash D) \dot{\cup} (A \cap C) \times (D \backslash B)$. You can rewrite them as a finite disjoint union. Also, $(A \times B) \cap (C \times D) = (A \cap C) \times (B \cap D)$, and $(A \times B)^c = (A^c \times Y) \dot{\cup} (A \times B^c)$. Thus, $\mathcal{A}$ is an algebra, and $\mathcal{A} \subseteq \mathcal{C}$.

Claim: $\mathcal{C}$ is a monotone class.

Proof of Claim: Suppose that $E_j \in \mathcal{C}$, and $E_j \subseteq E_{j+1} \subseteq \cdots$, and $E = \bigcup_j E_j$. Then, $(E_j)_x \subseteq (E_{j+1})_x \subseteq \cdots$, and $E_x = \bigcup_j (E_j)_x \Rightarrow \nu(E_x) = \lim_j \nu((E_j)_x) \Rightarrow x \to \nu(E_x)$ is $\mathcal{M}$-measurable. Similarly, $y \to \mu(E^y)$ is $\mathcal{N}$-measurable. Also, if $E_j \in \mathcal{C}$ and $E_j \supseteq E_{j+1} \supseteq \cdots$, and $E = \bigcap_j E_j$, then $(E_j)_x \supseteq (E_{j+1})_x \supseteq \cdots$, and $E_x = \bigcap_j (E_j)_x \Rightarrow \nu(E_x) = \lim_j \nu((E_j)_x)$ since ν is finite. Thus, (1) holds.

Similarly, (2) holds, and so $\mathcal{C}$ is a monotone class.

So, by the Monotone ~~Convergence~~ Class Theorem, the σ-algebra generated by $\mathcal{A} \subseteq \mathcal{C} \Rightarrow \mathcal{M} \otimes \mathcal{N} \subseteq \mathcal{C}$. So, every $E \in \mathcal{M} \otimes \mathcal{N}$ has properties (1) and (2).

[Show that $\mu \times \nu(E) = \int_X \nu(E_x) d\mu(x)$ is a measure.]

Suppose that $E = \dot{\bigcup}^{n} E_n$ and $E_n \in \mathcal{M} \otimes \mathcal{N}$.

Then, $E_x = \dot{\bigcup}^{n} (E_n)_x \Rightarrow \nu(E_x) = \sum_{n=1}^{\infty} \nu((E_n)_x)$.

So, $\mu \times \nu(E) = \int_X \nu(E_x) d\mu(x) = \int_X \sum_{n=1}^{\infty} \nu((E_n)_x) d\mu(x)$

$= \sum_{n=1}^{\infty} \int_X \nu((E_n)_x) d\mu(x) = \sum_{n=1}^{\infty} \mu \times \nu(E_n)$.

Proof for σ-finite case: Let $(X, \mathcal{M}, \mu)$ and $(Y, \mathcal{N}, \nu)$ be σ-finite. Then, $X = \dot{\bigcup}_{i=1}^{\infty} X_i$ and $\mu(X_i) < +\infty$ for each i. Also, $Y = \dot{\bigcup}_{j=1}^{\infty} Y_j$ and $\nu(Y_j) < +\infty$ for each j. If $E \in \mathcal{M} \otimes \mathcal{N}$, then E can be written as $\dot{\bigcup}_{i,j=1}^{\infty} [E \cap (X_i \times Y_j)]$, call these $E_{i,j} \subseteq X_i \times Y_j$.

By finite case, each $x \to \nu((E_{i,j})_x)$ and $y \to \mu((E_{i,j})^y)$ are measurable functions, $\nu(E_x) = \sum\limits_{i,j}\nu((E_{i,j})_x)$, and $\mu(E^y) = \sum\limits_{i,j}\mu((E_{i,j})^y)$. Also, $\int_X \nu(E_x)d\mu(x) = \sum\limits_{i,j}\int_X \nu((E_{i,j})_x)d\mu(x)$
$= \sum\limits_{i,j}\int_Y \mu((E_{i,j})^y)d\nu(y) = \int_Y \mu(E^y)d\nu(y)$.

<u>General Product Measure Non σ-finite Case</u>:

Given $(X, \mathcal{M}, \mu)$ and $(Y, \mathcal{N}, \nu)$, measure spaces, start with an outer measure on $X \times Y$ defined by $\lambda^*(E) = inf\{\sum\limits_j \mu(A_j)\nu(B_j) : E \subseteq \bigcup\limits_j A_j \times B_j,\ A_j \in \mathcal{M}$ and $B_j \in \mathcal{N}\}$. It is easy to show that λ^* is an outer measure. It is harder to show that every set in $\mathcal{M} \otimes \mathcal{N}$ is λ^*-measurable. To do this, use theorems about measures on algebra and show that $A \in \mathcal{M}$ and $B \in \mathcal{N} \Rightarrow \lambda^*(A \times B) = \mu(A)\nu(B)$. This always defines a measure $\mu \times \nu$. In fact, when they are both σ-finite measures, this measure is the same as the one given by Theorem 2.36.

Example (The reason why Theorem 2.36 requires σ-finite.)

Let $X = Y = [0,1]$, $\mathcal{M} = \mathcal{N} = \mathcal{B}_{\mathbb{R}}$, $\mu =$ Lebesgue measure which is finite ($\Rightarrow \sigma$-finite), and $\nu =$ counting measure which is <u>not</u> σ-finite. Let $D = \{(t,t) : 0 \le t \le 1\} \subseteq [0,1] \times [0,1]$ which is a diagonal of the square. Then, $D \in \mathcal{B}_{\mathbb{R}} \times \mathcal{B}_{\mathbb{R}} = \mathcal{B}_{\mathbb{R}^2}$, $D_x = \{x\}$, $\nu(D_x) = 1$, $D^y = \{y\}$ and $\mu(D^y) = 0$. Thus, $\int_X \nu(D_x)d\mu(x) = \int_{[0,1]} 1 d\mu = 1$ and $\int_Y \mu(D_y)d\nu(y) = \int_{[0,1]} 0 d\nu = 0$; they are not equal as in σ-finite case. Let $D \subseteq A_j \times B_j$ and $C_j = A_j \cap B_j$. Then, $D \subseteq \bigcup\limits_j (C_j \times C_j) \Rightarrow$ $[0,1] \subseteq \bigcup\limits_j C_j$ which is a countable union and $C_j \subset \mathcal{B}_{\mathbb{R}}$ covering $[0,1]$
$\Rightarrow$ there exists j_0 such that $\mu(C_{j_0}) > 0$, Lebesgue measure is positive
$\Rightarrow C_{j_0}$ is an infinite set $\Rightarrow \nu(C_{j_0}) = +\infty \Rightarrow \mu(C_{j_0})\nu(C_{j_0}) = +\infty$
$\Rightarrow \lambda^*(D) = +\infty$. Thus, $\mu \times \nu(D) = \ = +\infty$.

Note that $\int_X \nu(D_x)d\mu(x) = 1$ and $\int_Y \mu(D_y)d\nu(y) = 0$ are different, but finite. However, $\mu \times \nu(D) = \ = +\infty$ is nowhere near 0 or 1. So, when μ and ν are not both σ-finite, things get <u>bad</u>!