Claim: If $A \in \mathcal{M}$ and $B \in \mathcal{N}$, then $A \times B \in \mathcal{F}$.

Proof of Claim: $(A \times B)_x = \begin{cases} B & \text{if } x \in A \\ \emptyset & \text{if } x \notin A \end{cases}$ Since $B \in \mathcal{N}$ and $\emptyset \in \mathcal{N}$, $(A \times B)_x \in \mathcal{N}$ for all $x \in X$. Also,
$(A \times B)^y = \begin{cases} A & \text{if } y \in B \\ \emptyset & \text{if } y \notin B \end{cases} \Rightarrow (A \times B)^y \in \mathcal{M}$ for all $y \in Y$. Thus, $A \times B \in \mathcal{F}$ for all $A \in \mathcal{M}$ and for all $B \in \mathcal{N}$.

Thus, $\mathcal{M} \otimes \mathcal{N} \subseteq \mathcal{F}$, and hence $E \in \mathcal{M} \otimes \mathcal{N} \Rightarrow E \in \mathcal{F}$, that is $E_x \in \mathcal{N}$ for all $x \in X$ and $E^y \in \mathcal{M}$ for all $y \in Y$.

Proof of (b): Consider $f_{x_0}^{-1}((\alpha, +\infty]) = \{y : f_{x_0}(y) > \alpha\} = \{(x, y) : f(x, y) > \alpha\}_{x_0}$. Since f is measurable, $\{(x, y) : f(x, y) > \alpha\} \in \mathcal{M} \otimes \mathcal{N} \Rightarrow \{(x, y) : f(x, y) > \alpha\}_{x_0} = f_{x_0}^{-1}((\alpha, +\infty]) \in \mathcal{N}$
Thus, f_x is $\mathcal{N}$-measurable. Similarly, f^y is $\mathcal{M}$-measurable.

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Definition: Let X be a set. Then, $\mathcal{C} \subseteq \mathcal{P}(X)$ is called a **monotone class** if $E_j \in \mathcal{C}$ and $E_1 \subseteq E_2 \subseteq \cdots$, then $\bigcup_{j=1}^{\infty} E_j \in \mathcal{C}$, and also if $E_j \in \mathcal{C}$ and $E_1 \supseteq E_2 \supseteq \cdots$, then $\bigcap_{j=1}^{\infty} E_j \in \mathcal{C}$.

Note: Given any collection $\mathcal{E} \subseteq \mathcal{P}(X)$ of subsets, we can talk about the monotone class generated by $\mathcal{E}$

~~Folland;~~ Prop: ~~**Exercise 4 in page 24**~~: An algebra $\mathcal{A}$ is a σ-algebra if and only if $\mathcal{A}$ is closed under countable increasing unions (that is, if $\{E_j\}_{j=1}^{\infty} \subset \mathcal{A}$ and $E_1 \subset E_2 \subset \cdots$, then $\bigcup_{j=1}^{\infty} E_j \in \mathcal{A}$.)

Proof ($\Rightarrow$): Suppose that an algebra $\mathcal{A}$ is a σ-algebra, and suppose that $\{E_j\}_{j=1}^{\infty} \subset \mathcal{A}$ and $E_1 \subset E_2 \subset \cdots$. Then, $\bigcup_{j=1}^{\infty} E_j \in \mathcal{A}$ by the definition of σ-algebra.

Proof ($\Leftarrow$): Suppose that $\mathcal{A}$ is an algebra, and $\mathcal{A}$ is closed under countable increasing unions. Let $\{A_n\}_{n=1}^{\infty} \subset \mathcal{A}$, and let $B_1 = A_1$, $B_2 = A_1 \cup A_2$, $\cdots$. Then, $\bigcup_{n=1}^{\infty} A_n = \bigcup_{n=1}^{\infty} B_n$, and also $B_1 \subset B_2 \subset \cdots$. Thus, $\bigcup_{n=1}^{\infty} B_n \in \mathcal{A}$ and so $\bigcup_{n=1}^{\infty} A_n \in \mathcal{A}$.
Therefore, $\mathcal{A}$ is a σ-algebra.

Monotone Class Theorem 2.35: Let X be a nonempty set, and $\mathcal{A} \subseteq \mathcal{P}(X)$ be an algebra. Then, the monotone class generated by $\mathcal{A}$ is equal to the σ-algebra generated by $\mathcal{A}$.

Proof: Let X be a nonempty set, and $\mathcal{A} \subseteq \mathcal{P}(X)$ be an algebra. Also, let $\mathcal{C}$ be the monotone class generated by $\mathcal{A}$, and $\mathcal{M}$ be the σ-algebra generated by $\mathcal{A}$. Since every σ-algebra is a monotone class, $\mathcal{C} \subseteq \mathcal{M}$.

[Show that $\mathcal{M} \subseteq \mathcal{C}$. It is enough to show that $\mathcal{C}$ is a σ-algebra.]

Fix $E \in \mathcal{C}$, and let $\mathcal{C}_E = \{F \in \mathcal{C} : F \backslash E, E \backslash F, E \cap F \in \mathcal{C}\}$. It is clear that $\emptyset \in \mathcal{C}_E$ and $E \in \mathcal{C}_E$. Now, $F \in \mathcal{C}_E$ if and only if $E \in \mathcal{C}_F$.

[Show that $\mathcal{C}_E$ is a monotone class.]

Let $F_j \in \mathcal{C}_E$, $F_1 \subseteq F_2 \subseteq \cdots$, and $F = \bigcup_j F_j$. We know that $F \in \mathcal{C}$, $F_j \backslash E \in \mathcal{C}$ for all j, and $F_j \backslash E \subseteq F_{j+1} \backslash E$ (increasing.) This implies that $\bigcup_j (F_j \backslash E) = F \backslash E \in \mathcal{C}$. Next, $E \cap F_j \in \mathcal{C}$ for all j, and $E \cap F_j \subseteq E \cap F_{j+1}$ (increasing) implies that $\bigcup_j (E \cap F_j) = E \cap F \in \mathcal{C}$. Also, $E \backslash F_j \in \mathcal{C}$ for all j, and $E \backslash F_j \supseteq E \backslash F_{j+1}$ (decreasing) implies that $\bigcap_j (E \backslash F_j) = E \backslash F \in \mathcal{C}$. Thus, $F = \bigcup_j F_j \in \mathcal{C}_E$. Similarly, if $F_j \in \mathcal{C}_E$ and $F_j \supseteq F_{j+1}$, then $\bigcap_j F_j \in \mathcal{C}_E$. Therefore, $\mathcal{C}_E$ is a monotone class.

[Show that $\mathcal{C}_E = \mathcal{C}$.]

If $E, F \in \mathcal{A}$, then $F \in \mathcal{C}_E \Rightarrow \mathcal{A} \subseteq \mathcal{C}_E$ whenever $E \in \mathcal{A} \Rightarrow \mathcal{C} \subseteq \mathcal{C}_E \Rightarrow \mathcal{C} = \mathcal{C}_E$ whenever $E \in \mathcal{A}$.

[Show that $\mathcal{C}$ is an algebra.]

Since $X \in \mathcal{A}$, $\mathcal{C} = \mathcal{C}_X$. Thus, $E \in \mathcal{C} \Rightarrow E \in \mathcal{C}_X \Rightarrow X \in \mathcal{C}_E$ for any $E \in \mathcal{C}$. So, if $E \in \mathcal{C}$, then $X \backslash E = E^c \in \mathcal{C}$. Also, if $E, F \in \mathcal{C}$, then $E \backslash F, F \backslash E, E \cap F \in \mathcal{C}$. Thus, $\mathcal{C}$ is an algebra.

[Show that $\mathcal{C}$ is a σalgebra, and thus $\mathcal{M} \subseteq \mathcal{C}$.]

By Exercise 4 in page 24, $\mathcal{C}$ is a σ-algebra. Thus, $\mathcal{M} \subseteq \mathcal{C}$ and so $\mathcal{M} = \mathcal{C}$.

Theorem 2.36: Let $(X, \mathcal{M}, \mu)$ and $(Y, \mathcal{N}, \nu)$ be σ-finite measure spaces. If $E \in \mathcal{M} \otimes \mathcal{N}$, then:

(1) $x \to \nu(E_x)$ is $\mathcal{M}$-measurable, and $y \to \mu(E^y)$ is $\mathcal{N}$-measurable.

(2) $\int_X \nu(E_x) d\mu(x) = \int_Y \mu(E^y) d\nu(y)$

(3) If we set $\mu \times \nu(E) = \int_X \nu(E_x) d\mu(x)$, then $\mu \times \nu$ is a measure on $\mathcal{M} \otimes \mathcal{N}$.

Proof: First assume that $\nu(Y) < +\infty$ and $\mu(X) < +\infty$, and let $\mathcal{C} = \{E \in \mathcal{M} \otimes \mathcal{N} : \text{(1) and (2) hold.}\}$

<u>Claim</u>: If $E = A \times B$, $A \in \mathcal{M}$ and $B \in \mathcal{N}$, then $E \in \mathcal{C}$.

Proof of Claim: Let $E = A \times B$, $A \in \mathcal{M}$ and $B \in \mathcal{N}$. Then,

$E_x = (A \times B)_x = \begin{cases} B & \text{if } x \in A \\ \emptyset & \text{if } x \notin A \end{cases}$ and so $x \to \nu(E_x)$ $= \begin{cases} \nu(B) & \text{if } x \in A \\ 0 & \text{if } x \notin A \end{cases} = \nu(B)\chi_A(x)$. Also, $y \to \mu(E^y)$ $= \mu(A)\chi_B(y)$. Thus, (1) hold for $E = A \times B$, $A \in \mathcal{M}$ and $B \in \mathcal{N}$. Now, if $E = A \times B$, then $\int_X \nu(E_x) d\mu(x)$ $= \int_X \nu(B)\chi_A d\mu(x) = \nu(B)\mu(A)$ and $\int_Y \mu(E^y) d\nu(y)$ $= \int_Y \mu(A)\chi_B d\nu = (y)\mu(A)\nu(B)$. Thus, (2) also hold for $E = A \times B$, and so $E = A \times B \in \mathcal{C}$.

Next, suppose that $E =$ a finite disjoint union of rectangles $= \dot{\bigcup}_{j=1}^{n} (A_j \times B_j)$. Then, $E_x = \dot{\bigcup}_{j=1}^{n} (A_j \times B_j)_x$ is still a finite disjoint union $\Rightarrow \nu(E_x) = \nu(\dot{\bigcup}_{j=1}^{n} (A_j \times B_j)_x) = \sum_{j=1}^{n} \nu(A_j \times B_j)_x$ is measurable. Thus, (1) holds for $E = \dot{\bigcup}_{j=1}^{n} (A_j \times B_j)$. Also, when $E = \dot{\bigcup}_{j=1}^{n} (A_j \times B_j)$, $\nu(E_x) = \sum_{j=1}^{n} \nu(A_j \times B_j)_x = \sum_{j=1}^{n} \nu(B_j)\chi_{A_j}(x)$

$\Rightarrow \int_X \nu(E_x) d\mu(x) = \int_X \sum_{j=1}^{n} \nu(B_j)\chi_{A_j} d\mu(x,) = \sum_{j=1}^{n} \mu(A_j)\nu(B_j)$.

Similarly, if $E = \dot{\bigcup}_{j=1}^{n} (A_j \times B_j)$, then $\mu(E^y) = \sum_{j=1}^{n} \mu(A_j \times B_j)^y$ $= \sum_{j=1}^{n} \mu(A_j)\chi_{B_j}(y) \Rightarrow \int_Y \mu(E^y) d\nu(y) = \int_Y \sum_{j=1}^{n} \mu(A_j)\chi_{B_j} d\nu(y)$ $= \sum_{j=1}^{n} \mu(A_j)\nu(B_j)$. Thus, (2) holds, and so $E = \dot{\bigcup}_{j=1}^{n} (A_j \times B_j)$ $\in \mathcal{C}$.