Examples:

(i) Let $f_n = 1/n\chi_{(0,n)}$. Then:
$f_n \to f$ in measure since $\{x : |f_n - 0| \geq \epsilon\} = \emptyset$ for $1/n < \epsilon$.

(iii) Let $f_n = n\chi_{[0,1/n]}$. Then:
$f_n \to 0$ in measure since $\{x : |f_n(x) - 0| \geq \epsilon\} \subseteq [0, 1/n]$.

(iv) Let $f_n = \chi_{[j/2^k,(j+1)/2^k]}$ where $n = 2^k + j$ and $0 \leq j \leq 2^k$. That is,
$f_1 = \chi_{[0,1]}, f_2 = \chi_{[0,1/2]}, f_3 = \chi_{[1/2,1]}, f_4 = \chi_{[0,1/4]}, f_5 = \chi_{[1/4,1/2]}$
$f_6 = \chi_{[1/2,3/4]}, f_7 = \chi_{[3/4,1]}, f_8 = \chi_{[0,1/8]}, \cdots$ Then:
$f_n \to 0$ in measure.

~~END~~ HER

Proposition 2.29: If $\int_X |f_n - f| d\mu \to 0$, that is $f_n \to f$ in L^1, then $f_n \to f$ in measure.

Proof: Given $\epsilon > 0$, let $E_{n,\epsilon} = \{x : |f_n - f| \geq \epsilon\}$. Then, $0 \leftarrow \int_X |f_n - f| d\mu \geq \int_{E_{n,\epsilon}} |f_n - f| d\mu \geq \epsilon\mu(E_{n,\epsilon}) \Rightarrow \lim_n \mu(E_{n,\epsilon}) = 0$ for all $\epsilon > 0$.

<u>Note</u> that the converse of Proposition 2.29 is false: Let $f_n = 1/n\chi_{(0,n)}$. Then, $f_n \to 0$ in measure, but $\int_{\mathbb{R}} |f_n - 0| dm = 1 \not\to 0$.

END HERE → START 2/13

Theorem 2.30: Let $(X, \mathcal{M}, \mu)$ be a measure space, and $f_n : X \to \mathbb{R}$ be measurable. Then, if $f_n \to f$ in measure and $f_n \to g$ in measure, then $f = g$ *a.e.* If $\{f_n\}$ is Cauchy in measure, then there exists $f : X \to \mathbb{R}$ which is measurable such that $f_n \to f$ in measure. Moreover, if $f_n \to f$ in measure, then there exists a subsequence $\{f_{n_j}\}$ such that $f_{n_j} \to f$ pointwise *a.e.*

[Note that $f_n \to f$ in measure $\Leftrightarrow$ $\{f_n\}$ is Cauchy in measure.]

Proof: Let $(X, \mathcal{M}, \mu)$ be a measure space, and $f_n : X \to \mathbb{R}$ be measurable. Suppose that $f_n \to f$ in measure and $f_n \to g$ in measure. Then, given $\epsilon > 0$, $\{x : |f(x) - g(x)| \geq \epsilon\} \subseteq \{x : |f(x) - f_n(x)| \geq \epsilon/2\} \cup \{x : |g(x) - f_n(x)| \geq \epsilon/2\}$ because $\epsilon \leq |f(x) - g(x)| \leq |f(x) - f_n(x)| + |f_n(x) - g(x)|$. So, $\mu(\{x : |f(x) - g(x)| \geq \epsilon\}) \leq \mu(\{x : |f(x) - f_n(x)| \geq \epsilon/2\}) + \mu(\{x : |g(x) - f_n(x)| \geq \epsilon/2\})$ for all n, and also by the hypotheses $\mu(\{x : |f(x) - f_n(x)| \geq \epsilon/2\}) \to 0$ and $\mu(\{x : |g(x) - f_n(x)| \geq \epsilon/2\}) \to 0$ as $n \to \infty$. Thus, $\mu(\{x : |f(x) - g(x)| \geq \epsilon\}) = 0$ for all $\epsilon > 0$.

Now, $\{x : f(x) \neq g(x)\} = \bigcup_{n=1}^{\infty} \{x : |f(x) - g(x)| \geq 1/n\} \Rightarrow \mu(\{x : f(x) \neq g(x)\}) = 0$. Thus, $f = g$ *a.e.*

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Assume that $\{f_n\}$ is Cauchy in measure. Let $\epsilon_1 = \epsilon_2 = 1/2$, then there exists N_1 such that $n, m \geq N_1$ implies that $\mu(\{x : |f_n - f_m| \geq 1/2\}) < 1/2$. Let $g_1 = f_{N_1}$. Now, pick $N_2 > N_1$ such that $\epsilon_1 = \epsilon_2 = 1/4$. Then, whenever $n, m \geq N_2$, $\mu(\{x : |f_n - f_m| \geq 1/4\}) < 1/4$. Let $g_2 = f_{N_2}$.
Continue the process inductively to get $g_j = f_{N_j}$ such that

$\mu(E_j) = \mu(\{x : |g_{j+1} - g_j| \geq 1/2^j\}) < 1/2^j$. Let $F_k = \bigcup_{j=k}^{\infty} E_j$

(the tail end of E_j's), then $\mu(F_k) \leq \sum_{j=k}^{\infty} \mu(E_j) < \sum_{j=k}^{\infty} 2^{-j} = 2^{1-k}$.

Let $F_\infty = \bigcap_{k=1}^{\infty} F_k$. Remember that $F_1 \supseteq F_2 \supseteq \cdots$. Next, let

$h(x) = g_1(x) + \sum_{l=1}^{\infty}(g_{l+1}(x) - g_l(x))$. If $x \notin F_k$ (that is, $x \in F_k^c$), $x \notin E_j$ for all $j \geq k \Rightarrow |g_{j+1} - g_j| < 1/2^j \Rightarrow$ the above series is absolutely convergent if $x \in F_k^c$, which is true for all k.

Thus, the series for h is absolutely convergent for all $x \in \bigcup_{k=1}^{\infty} F_k^c$

$= (\bigcap_{k=1}^{\infty} F_k)^c = F_\infty^c$. Hence, if $x \notin F_\infty$, $h(x) = \lim_j [g_1(x) +$

$\sum_{l=1}^{j}(g_{l+1}(x) - g_l(x))] = \lim_j g_{j+1}(x)$. Thus, $g_j(x) = f_{N_j}(x) \to$ $h(x)$ for all $x \notin F_\infty$. But, $\mu(F_\infty) = \lim_k \mu(F_k) = 0$. Therefore,

$f_{N_j} \to h$ pointwise *a.e.*

Corollary 2.32: If $\int_X |f_n - f| d\mu \to 0$, then there exists $\{f_{n_j}\}$ such that $f_{n_j} \to f$ pointwise *a.e.*

Proof: By Proposition 2.29, $\int_X |f_n - f| d\mu \to 0 \Rightarrow f_n \to f$ in measure ~~$\Rightarrow$ every subsequence $f_{n_j} \to f$ in measure. But, by Theorem~~
By Theorem 2.30, we can pick $\{f_{n_j}\}$ such that $f_{n_j} \to \cancel{f}$ pointwise *a.e.*
~~[Check] $f_{n_j} \to h$ in measure $\Rightarrow f = h$ *a.e.* Thus, $f_{n_j} \to f$ pointwise *a.e.*~~

Recall example (iv)

Egoroff's Theorem: Assume that $\mu(X) < +\infty$. Let $f_n \to f$ pointwise *a.e.* Then, given $\epsilon > 0$, there exists $E \subseteq X$ and $\mu(E) < \epsilon$ such that $f_n \to f$ uniformly on E^c.

Note: A sequence of functions may converges to a function pointwise, but it is not guaranteed that the function is continuous. But, Egoroff's Theorem says that if we throw away a not-nice part, even a crazy or wild sequence of functions converges pointwise to a function which is continuous.

Proof: Write $X = X_1 \cup N$ where $\mu(N) = 0$ and $f_n(x) \to f(x)$ for all $x \in X_1$. Let $E_{n,k} = \bigcup_{m=n}^{\infty} \{x \in X_1 : |f_m(x) - f(x)| \geq 1/k\}$.

Note that $E_{n,k} \supseteq E_{n+1,k} \supseteq \cdots$. So, $\bigcap_{n=1}^{\infty} E_{n,k} = \emptyset \Rightarrow \lim_n \mu(E_{n,k}) = 0$. Pick n_k such that $\mu(E_{n_k,k}) < \epsilon/2^k$, and let $E = \bigcup_{k=1}^{\infty} E_{n_k,k} \Rightarrow \mu(E) \leq \sum_{k=1}^{\infty} \mu(E_{n_k,k}) < \epsilon$. Pick $\delta > 0$ such that $1/K < \delta$. Then, $x \in E^c \Rightarrow x \notin E \Rightarrow$ x $\notin E_{n_K,K}$ for all $K \Rightarrow |f_m(x) - f(x)| \leq 1/K < \delta$ for any $m > n_K$. Thus, $f_n \to f$ uniformly on E^c.

– Midterm Material Ends Here ————————

Midterm: Feb. 27, in-class

Final: Sat. April 21, 4-6:30 pm, PAC Upper 11

<u>HW 11:</u> Assume $\mu(X) < +\infty$

Let $Y = \{ [f] \mid f: X \to \mathbb{R}$ measurable, $[f] = [g] \Leftrightarrow f = g$ a.e.$\}$.

Prove that:

(1) $\rho([f],[g]) = \int_X \frac{|f-g|}{1+|f-g|} \, d\mu$ is a metric on Y

(2) $[f_n], [f] \in Y$ then $\rho([f],[f_n]) \to 0$ $\Longleftrightarrow$ $f_n \to f$ in μ-measure

(Hint for (1): Show $0 \le a \le b + c \Rightarrow \frac{a}{1+a} \le \frac{b}{1+b} + \frac{c}{1+c}$)

<u>HW 12:</u> Let f_n, f be measurable, $f_n \geq 0$

If $f_n \to f$ in measure, then

$$\int f \, d\mu \le \liminf_n \int f_n \, d\mu$$

2.5 Product Measures

<u>Goals</u>:
(1) Suppose that we are given measure spaces $(X, \mathcal{M}, \mu)$ and $(Y, \mathcal{N}, \nu)$. We want a measure on the product $X \times Y$, $\mu \times \nu$ such that $\mu \times \nu(A \times B) = \mu(A) \cdot \nu(B)$ where $A \in \mathcal{M}$ and $B \in \mathcal{N}$. We want to construct something like this in a general product space.
(2) In Calculus, if $I = [a, b] \times [c.d]$, then $\int_I f dA = \int_a^b \int_c^d f(x, y) dy dx = \int_c^d \int_a^b f(x, y) dx dy$. We want to prove a formal theorem to justify the above.

<u>Recall</u> that given measurable spaces $(X, \mathcal{M})$ and $(Y, \mathcal{N})$, the σ-algebra of subsets of $X \times Y$ generated by the sets of the form $\{A \times B : A \in \mathcal{M}$ and $B \in \mathcal{N}\}$ is denoted by $\mathcal{M} \otimes \mathcal{N}$ and called the **product σ-algebra.**

Definition: Suppose that $E \subseteq X \times Y$. For any $x \in X$, we define $E_x = \{y : (x, y) \in E\}$. Also, for any $y \in Y$, we define $E^y = \{x : (x, y) \in E\}$. If $f : X \times Y \to \overline{\mathbb{R}}$, then we define $f_x : Y \to \overline{\mathbb{R}}$ by $f_x(y) = f(x, y)$ for any $x \in X$, and $f^y : X \to \overline{\mathbb{R}}$ by $f^y(x) = f(x, y)$ for any $y \in Y$.

Proposition 2.34: Let $(X, \mathcal{M})$ and $(Y, \mathcal{N})$ be measurable spaces. Then:
(a) If $E \in \mathcal{M} \otimes \mathcal{N}$, then $E_x \in \mathcal{N}$ for all $x \in X$, and $E^y \in \mathcal{M}$ for all $y \in Y$.
(b) If $f : X \times Y \to \overline{\mathbb{R}}$ is $\mathcal{M} \otimes \mathcal{N}$-measurable, then $f_x : Y \to \overline{\mathbb{R}}$ is $\mathcal{N}$-measurable for $x \in X$, and $f^y : X \to \overline{\mathbb{R}}$ is $\mathcal{M}$-measurable for all $y \in Y$.

Proof of (a): Let $\mathcal{F}$ be the set $\{E \subseteq X \times Y : E_x \in \mathcal{N}$ for all $x \in X$ and $E^y \in \mathcal{M}$ for all $y \in Y\}$.

<u>Claim</u>: $\mathcal{F}$ is a σ-algebra.

Proof of Claim: It is clear that $\emptyset, X \times Y \in \mathcal{F}$. If $E \in \mathcal{F}$, then $(E^c)_x = (E_x)^c \in \mathcal{N}$ because $E_x \in \mathcal{N}$ and $\mathcal{N}$ is a σ-algebra. Similarly, $(E^c)^y = (E^y)^c \in \mathcal{M}$. If $E_n \in \mathcal{F}$, then $(\bigcup_{n=1}^{\infty} E_n)_x = \bigcup_{n=1}^{\infty} (E_n)_x \in \mathcal{N}$. Similarly, $(\bigcup_{n=1}^{\infty} E_n)^y = \bigcup_{n=1}^{\infty} (E_n)^y \in \mathcal{M}$. Thus, $\mathcal{F}$ is a σ-algebra.

Claim: If $A \in \mathcal{M}$ and $B \in \mathcal{N}$, then $A \times B \in \mathcal{F}$.

Proof of Claim: $(A \times B)_x = \begin{cases} B & \text{if } x \in A \\ \emptyset & \text{if } x \notin A \end{cases}$ Since $B \in \mathcal{N}$ and $\emptyset \in \mathcal{N}$, $(A \times B)_x \in \mathcal{N}$ for all $x \in X$. Also, $(A \times B)^y = \begin{cases} A & \text{if } y \in B \\ \emptyset & \text{if } y \notin B \end{cases} \Rightarrow (A \times B)^y \in \mathcal{M}$ for all $y \in Y$. Thus, $A \times B \in \mathcal{F}$ for all $A \in \mathcal{M}$ and for all $B \in \mathcal{N}$.

Thus, $\mathcal{M} \otimes \mathcal{N} \subseteq \mathcal{F}$, and hence $E \in \mathcal{M} \otimes \mathcal{N} \Rightarrow E \in \mathcal{F}$, that is $E_x \in \mathcal{N}$ for all $x \in X$ and $E^y \in \mathcal{M}$ for all $y \in Y$.

Proof of (b): Consider $f_{x_0}^{-1}((\alpha, +\infty]) = \{y : f_{x_0}(y) > \alpha\} = \{(x, y) : f(x, y) > \alpha\}_{x_0}$. Since f is measurable, $\{(x, y) : f(x, y) > \alpha\} \in \mathcal{M} \otimes \mathcal{N} \Rightarrow \{(x, y) : f(x, y) > \alpha\}_{x_0} = f_{x_0}^{-1}((\alpha, +\infty]) \in \mathcal{N}$

Thus, f_x is $\mathcal{N}$-measurable. Similarly, f^y is $\mathcal{M}$-measurable.

Definition: Let X be a set. Then, $\mathcal{C} \subseteq \mathcal{P}(X)$ is called a **monotone class** if $E_j \in \mathcal{C}$ and $E_1 \subseteq E_2 \subseteq \cdots$, then $\bigcup_{j=1}^{\infty} E_j \in \mathcal{C}$, and also if $E_j \in \mathcal{C}$ and $E_1 \supseteq E_2 \supseteq \cdots$, then $\bigcap_{j=1}^{\infty} E_j \in \mathcal{C}$.

Note: Given any collection $\mathcal{E} \in \mathcal{P}(X)$ of subsets, we can talk about the monotone class generated by $\mathcal{E}$

Folland: **Exercise 4 in page 24**: An algebra $\mathcal{A}$ is a σ-algebra if and only if $\mathcal{A}$ is closed under countable increasing unions (that is, if $\{E_j\}_{j=1}^{\infty} \subset \mathcal{A}$ and $E_1 \subset E_2 \subset \cdots$, then $\bigcup_{j=1}^{\infty} E_j \in \mathcal{A}$.)

Proof ($\Rightarrow$): Suppose that an algebra $\mathcal{A}$ is a σ-algebra, and suppose that $\{E_j\}_{j=1}^{\infty} \subset \mathcal{A}$ and $E_1 \subset E_2 \subset \cdots$. Then, $\bigcup_{j=1}^{\infty} E_j \in \mathcal{A}$ by the definition of σ-algebra.

Proof ($\Leftarrow$): Suppose that $\mathcal{A}$ is an algebra, and $\mathcal{A}$ is closed under countable increasing unions. Let $\{A_n\}_{n=1}^{\infty} \subset \mathcal{A}$, and let $B_1 = A_1$, $B_2 = A_1 \cup A_2$, $\cdots$. Then, $\bigcup_{n=1}^{\infty} A_n = \bigcup_{n=1}^{\infty} B_n$, and also $B_1 \subset B_2 \subset \cdots$. Thus, $\bigcup_{n=1}^{\infty} B_n \in \mathcal{A}$ and so $\bigcup_{n=1}^{\infty} A_n \in \mathcal{A}$. Therefore, $\mathcal{A}$ is a σ-algebra.