Difference between Riemann Integration and Lebesgue Integration

Recall: Let $f : [a, b] \to \mathbb{R}$ be bounded, $P = \{t_0, t_1, \cdots, t_n\}$ where $a = t_0 < t_1 < \cdots < t_n = b$. P' refines P if $P \subseteq P'$. Given P, let $M_j = sup\{f(t) : t_{j-1} \le t \le t_j\}$ and $m_j = inf\{f(t) : t_{j-1} \le t \le t_j\}$. Then, $U_P f = \sum_{j=1}^{n} M_j(t_j - t_{j-1})$ and $L_P f = \sum_{j=1}^{n} m_j(t_j - t_{j-1})$. Define $\overline{I}_a^b f = \inf_P U_P f$ which is called the **upper Riemann Integral**, and $\underline{I}_a^b f = \sup_P L_P f$ which is called the **lower Riemann Integral**. f is called **Riemann Integrable** if $\overline{I}_a^b f = \underline{I}_a^b f$. In this case, we write $\int_a^b f(x)dx$ for their common value.

Definition: Let $f : [a, b] \to \mathbb{R}$. Then, $U(x) = \lim_{\delta \downarrow 0} \sup_{|y-x| \le \delta} f(y) = \inf_{\delta > 0} \sup_{|y-x| \le \delta} f(y)$ is called the **upper envelope of f**. Note that $f(x) \le U(x)$, $\inf_{\delta > 0} \sup_{0 < |y-x| \le \delta} f(y) \equiv \overline{\lim_{y \to x}} f(y)$, and so $U(x) = max\{f(x), \overline{\lim_{y \to x}} f(y)\}$.
$L(x) = \lim_{\delta \downarrow 0} \inf_{|y-x| \le \delta} f(y) = \sup_{\delta > 0} \inf_{|y-x| \le \delta} f(y)$ is called the **lower envelope of f**. Note that $L(x) \le f(x)$, $\sup_{\delta > 0} \inf_{0 < |y-x| \le \delta} f(y) \equiv \underline{\lim_{y \to x}} f(y)$, and so $L(x) = min\{f(x), \underline{\lim_{y \to x}} f(y)\}$.

Exercise 23(a) in page 59: Let U be the upper envelope of f and L be the lower envelope of f. Then, $U(x) = L(x)$ if and only if f is continuous at x.

OVER for easier proof

Proof: Suppose that $U(x_0) < \alpha$, then there exists $\delta_0 > 0$ such that $\sup_{|x_0 - y| \le \delta_0} f(y) = \beta < \alpha$. Thus, $x_0 - \delta_0 \le y \le x_0 + \delta_0 \Rightarrow f(y) \le \beta$. So, if we pick y_0 such that $x_0 - \delta_0 < y_0 < x_0 + \delta_0$, then there exists δ_1 such that $x_0 - \delta_0 < y_0 - \delta_1 < y_0 < y_0 + \delta_1 < x_0 + \delta_0 \Rightarrow \sup_{|y-y_0| \le \delta_1} f(y) = \sup_{|y-x_0| \le \delta_0} f(y) = \beta < \alpha \Rightarrow U(y_0) < \beta$ $\Rightarrow U(y_0) < \alpha$ for all y_0 such that $x_0 - \delta_0 < y_0 < x_0 + \delta_0 \Rightarrow$ $\{x : U(x) < \alpha\} = U^{-1}((-\infty, \alpha))$ is open $\Rightarrow U$ is measurable and upper semicontinuous $\Leftrightarrow f(x) = U(x)$.
Similarly, $\{x : L(x) > \alpha\} = L^{-1}((\alpha, \infty))$ is open $\Rightarrow L$ is measurable and lower semicontinuous $\Leftrightarrow L(x) = f(x)$.
Thus, $L(x) = U(x) \Leftrightarrow f$ is continuous at x.

$f(x) \le U(x) = L(x) = \min\{f(x), -\} = f(x)$

$\therefore\ U(x) = L(x) \Rightarrow \quad U(x) = L(x) = f(x)$

$\overline{\lim\limits_{y\to x}}\ f(y) \le U(x) = f(x) = L(x) \le \underline{\lim\limits_{y\to x}}\ f(y)$

$\Rightarrow \overline{\lim} = \underline{\lim} \Rightarrow \lim\limits_{y\to x} f(y)$ exists and is $f(x)$

$\Rightarrow f$ cont at x

f cont at $x \Rightarrow \overline{\lim}\ f(y) = f(x) = \underline{\lim}$

$\Rightarrow \quad U(x) = L(x)$

Proposition: Let L and U be the lower and upper envelopes of f. Then there exist step functions $\{\phi_n\}$ such that $\phi_n \nearrow L$, and step functions $\{\psi_n\}$ such that $\psi_n \searrow U$. Hence, U, L measurable.

Proof: Let $P = \{a = t_0 < t_1 < \cdots < t_n = b\}$, and $m_i = inf\{L(t) : t_{j-1} \leq t \leq t_j\}$.

Now, define $\phi_P(x) = \begin{cases} m_i & \text{if } t_{i-1} < t < t_i \\ min\{m_i, m_{i+1}\} & \text{if } t = t_i \end{cases}$

Notice that $\phi_p(x) \leq L(x)$ for all x. Next, take the partition of $[a, b]$ into the pieces of the length $(b-a)/2^n$, call this partition P_n. Notice that $P_1 \subseteq P_2 \subseteq \cdots$. So, if we set $\phi_n = \phi_{P_n}$, then $\phi_1 \leq \phi_2 \leq \cdots \leq L(x)$ and given $\epsilon > 0$, there exists $\delta > 0$ such that $|L(x) - \inf_{|x-y|\leq\delta} f(y)| < \epsilon \Rightarrow L(x) \leq \inf_{|x-y|\leq\delta} f(y) + \epsilon$ $\Rightarrow L(x) \leq L(y) + \epsilon$ for $|x - y| < \delta$. Now, pick N such that $(b-a)/2^N < \delta$. Then, for all $n \geq N$, if $x, y \in [t_{i-1}, t_i]$, then $|x - y| < \delta$. This implies that $m_i = inf\{L(y) : t_{i-1} \leq y \leq t_i\}$ $\geq L(x) - \epsilon \Rightarrow \phi_n(y) \geq L(x) - \epsilon$ for all $n \geq N \Rightarrow \lim_n \phi_n(x)$ $= L(x)$ for all x.

Similarly, $\lim_n \psi_n(x) = U(x)$ for all x.

Theorem (Exercise 23(b) in page 59): Let $f : [a, b] \to \mathbb{R}$ be bounded, L be the lower envelope of f and U be the upper envelope of f. Then, $\overline{I}_a^b f = \int_{[a,b]} U dm$ and $\underline{I}_a^b f = \int_{[a,b]} L dm$.

Proof: Let $P = \{a = t_0 < t_1 < \cdots < t_n = b\}$, and define $G_P = \sum_{j=1}^{n} M_j \chi_{(t_{j-1}, t_j]}$ where $M_j = sup\{f(t) : t_{j-1} \leq t \leq t_j\}$.

Then, $x \in [t_{j-1}, t_j] \Rightarrow U(x) \leq G_p(x)$. Thus, if $U(x) \leq G_p(x)$ for all x, then $\int_{[a,b]} U dm \leq \inf_P \int_{[a,b]} G_P dm = \overline{I}_a^b f$. Similarly, $g_P \leq L \Rightarrow \underline{I}_a^b f = \sup_P \int_{[a,b]} g_P dm \leq \int_{[a,b]} L dm$.

Now, by the above Proposition, there exists $\phi_n \nearrow L$, each ϕ_n is a step function and $\phi_n \leq L \leq f \Rightarrow \underline{I}_a^b f \geq \int_{[a,b]} \phi_n dm$ and $\lim_n \int_{[a,b]} \phi_n dm = \int_{[a,b]} L dm$ by the Monotone Convergence Theorem $\Rightarrow \underline{I}_a^b f \geq \int_{[a,b]} L dm$, and so $\underline{I}_a^b f = \int_{[a,b]} L dm$.

Similarly, $\overline{I}_a^b f = \int_{[a,b]} U dm$.

Theorem 2.28: Let $f : [a, b] \to \mathbb{R}$ be bounded. Then:

(a) If f is Riemann integrable, then f is Lebesgue measurable (that is, $f^{-1}((a, +\infty))$ is a Lebesgue set for all a), and $\int_a^b f(x)dx = \int_{[a,b]} f dm$ where m is Lebesgue measure.

(b) f is Riemann integrable $\Leftrightarrow \{x \in [a, b] : f(x)$ is not continuous at $x\}$ has Lebesgue measure 0.

Proof of (a): Given a partition $P = \{a = t_0 < t_1 < \cdots < t_n = b\}$, define $G_P = \sum_{j=1}^{n} M_j \chi_{(t_{j-1},t_j]}$ and $g_P = \sum_{j=1}^{n} m_j \chi_{(t_{j-1},t_j]}$. Notice that $g_P \leq f \leq G_P$, and G_p and g_P are both simple. Also, note that if P' refines P, then $G_{P'} \leq G_P$ and $g_P \leq g_{P'}$.

First, pick a sequence of partitions, $P_1.P_2, \cdots$ so that $\overline{I}_a^b f = \lim_n U_{P_n} f$. Now, let $\widetilde{P_1} = P_1, \widetilde{P_2} = \widetilde{P_1} \cup P_2, \widetilde{P_n} = \widetilde{P_{n-1}} \cup P_n = P_1 \cup P_2 \cup \cdots \cup P_n$. Notice that each $\widetilde{P_n}$ is a refinement of $P_n \Rightarrow U_{\widetilde{P_n}} f \leq U_{P_n} f \Rightarrow \overline{I}_a^b f = \lim_n U_{\widetilde{P_n}} f$. Finally, $\widetilde{P_1} \subseteq \widetilde{P_2} \subseteq \cdots$ are all refinement. Thus, $G_{\widetilde{P_1}} \geq G_{\widetilde{P_2}} \geq \cdots \geq f$, and so note that for any partition, $U_P f = \sum M_j (t_j - t_{j-1}) = \int_{[a,b]} G_P dm$. So, let $G(x) = \inf_n G_{\widetilde{P_n}}(x) = \lim_n G_{\widetilde{P_n}}(x)$. Then, $G(x)$ is measurable (since the limit of measurable functions is measurable), $G(x) \geq f(x)$, and $\int_{[a,b]} G_P dm = \lim_n \int_{[a,b]} G_{\widetilde{P_n}} dm = \overline{I}_a^b f$ [like the Monotone Convergence Theorem, because the sequence is decreasing and the first one is finite.]

Do the same for $\underline{I}_a^b f$, pick $\widetilde{\widetilde{P_1}} \subseteq \widetilde{\widetilde{P_2}} \subseteq \cdots$ such that $\underline{I}_a^b f = \lim_n L_{\widetilde{\widetilde{P_n}}} f = \lim_n \int_{[a,b]} g_{\widetilde{\widetilde{P_n}}} dm = \int_{[a,b]} g dm$ (note that the last equality holds by the Monotone Convergence Theorem) where $g(x) = \sup_n g_{\widetilde{\widetilde{P_n}}}(x) = \lim_n g_{\widetilde{\widetilde{P_n}}}(x)$, and $g_{\widetilde{\widetilde{P_1}}}(x) \leq g_{\widetilde{\widetilde{P_2}}}(x) \leq \cdots \leq f(x)$. Thus, $g(x)$ is also measurable. Since f is Riemann integrable, $\int_{[a,b]} g dm = \underline{I}_a^b f = \overline{I}_a^b f = \int_{[a,b]} G_P dm \Rightarrow \int_{[a,b]} (G - g) dm = 0$, but $G - g \geq 0 \Rightarrow G - g = 0$ *a.e.* $\Rightarrow G = f$ *a.e.* (or $g = f$ *a.e.*) $\Rightarrow f$ is Lebesgue measurable. Also, $g \leq f \leq G \Rightarrow \int_a^b f(x)dx = \underline{I}_a^b f = \int_{[a,b]} g dm \leq \int_{[a,b]} f dm \leq \int_{[a,b]} G dm = \overline{I}_a^b f = \int_a^b f(x)dx \Rightarrow \int_{[a,b]} f dm = \int_a^b f(x)dx$.

Proof of (b): Suppose that f is Riemann integrable. Then, $\overline{I}_a^b f = \underline{I}_a^b f$
$\Rightarrow \int_{[a,b]} U dm = \int_{[a,b]} L dm$ by the Theorem (23(b)) above
$\Rightarrow \int_{[a,b]} (U - L) dm = 0 \Rightarrow U = L$ *a.e.* $\Rightarrow \{x : f$ is not continuous at $x\}$ has measure 0.
Conversely, suppose that $\{x : f$ is not continuous at $x\}$ has measure 0. Then, $\{x : U(x) \neq L(x)\}$ has measure 0 $\Rightarrow$ $U(x) = L(x)$ *a.e.* $\Rightarrow \overline{I}_a^b f = \int_{[a,b]} U dm = \int_{[a,b]} L dm = \underline{I}_a^b f$ by the Theorem (23(b)) above $\Rightarrow f$ is Riemann integrable.

Note that upper and lower envelopes of Lebesgue integrals are Riemann integrals.

Example: Let $f(x) = \begin{cases} 0 & \text{if } x \leq 0 \\ 1/\sqrt{x} & \text{if } x > 0 \end{cases}$ Compute $\int_{\mathbb{R}} f dm$.

Let $f_n(x) = \begin{cases} 0 & \text{if } x, 1/n \\ 1/\sqrt{x} & \text{if } 1/n \leq x \leq n \\ 0 & \text{if } x > n \end{cases}$ Then, $f_n \nearrow f$.

By the Monotone Convergence Theorem, $\int_{\mathbb{R}} f dm = \lim_n \int_{\mathbb{R}} f_n dm$.
Now, $\int_{\mathbb{R}} f_n dm = \int_{[1/n,n]} f_n dm = R\int_{1/n}^{n} 1/\sqrt{x} dx = 2\sqrt{x}\big]_{1/n}^{n} = 2\sqrt{n} - 2/\sqrt{n} \to +\infty$ as $n \to +\infty$. Thus, $\int_{\mathbb{R}} f dm = +\infty$.

2.4 Modes of Convergence

Meaning of $f_n \to f$

- $f_n \to f$ pointwise $\Leftrightarrow \lim_n f_n(x) = f(x)$ for all x
- $f_n \to f$ pointwise *a.e.* $\Leftrightarrow \lim_n f_n(x) = f(x)$ except for x in a set of measure 0.
- $f_n \to f$ uniformly on $E \Leftrightarrow$ For all $\epsilon > 0$, there exists N such that $|f_n(x) - f(x)| < \epsilon$ for all $n \geq N$ and for all $x \in E$.
- $f_n \to f$ in $L^1 \Leftrightarrow \int_X |f_n - f| d\mu \to 0$

Key Examples:

(i) Let $f_n = 1/n\chi_{(0,n)}$. Then:
$f_n \to 0$ uniformly, but $\int f_n d\mu = 1 \not\to \int 0 d\mu \Rightarrow \int |f_n - 0| d\mu \not\to 0 \Rightarrow$ $f_n \not\to 0$ in L^1.

(ii) Let $f_n = \chi_{(n,n+1)}$. Then:
$f_n \to 0$ pointwise, but $\int f_n d\mu = 1 \not\to \int 0 d\mu \Rightarrow \int |f_n - 0| d\mu \not\to 0 \Rightarrow$ $f_n \not\to 0$ in L^1.

(iii) Let $f_n = n\chi_{[0,1/n]}$. Then:
$f_n \to 0$ pointwise *a.e.* on $[0,1]$, but $\int_{[0,1]} f_n d\mu = 1 \not\to \int_{[0,1]} 0 d\mu \Rightarrow$ $\int_{[0,1]} |f_n - 0| d\mu \not\to 0 \Rightarrow f_n \not\to 0$ in L^1.

(iv) Let $f_n = \chi_{[j/2^k,(j+1)/2^k]}$ where $n = 2^k + j$ and $0 \leq j \leq 2^k$. That is,
$f_1 = \chi_{[0,1]}, f_2 = \chi_{[0,1/2]}, f_3 = \chi_{[1/2,1]}, f_4 = \chi_{[0,1/4]}, f_5 = \chi_{[1/4,1/2]}$
$f_6 = \chi_{[1/2,3/4]}, f_7 = \chi_{[3/4,1]}, f_8 = \chi_{[0,1/8]}, \cdots$
For any x, $f_n(x) = \begin{cases} 1 & \text{infinitely often} \\ 0 & \text{infinitely often} \end{cases}$
$\overline{\lim_n} f_n(x) = 1$ and $\overline{\lim_n} f_n(x) = \chi_{[0,1]}(x)$ $\quad$ f_n does not converge pointwise
$\underline{\lim_n} f_n(x) = 0$ and $\underline{\lim_n} f_n = 0$
$\int_{[0,1]} f_n d\mu \to 0 \Rightarrow f_n \to 0$ in L^1.

Definition: Let $(X, \mathcal{M}, \mu)$ be a measure space, and $f_n, f : X \to \mathbb{R}$ be measurable. Then, we say that **$\{f_n\}$ converges to f in measure** (that is, $f_n \to f$ in measure) provided that for all $\epsilon > 0$, $\mu(\{x : |f_n(x) - f(x)| \geq \epsilon\}) \to 0$ as $n \to \infty$.

We say that **$\{f_n\}$ is Cauchy in measure** if for all $\epsilon_1 > 0$ and $\epsilon_2 > 0$, there exists N such that if $n, m \geq N$, then $\mu(\{x : |f_n(x) - f_m(x)| \geq \epsilon_1\}) < \epsilon_2$.

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Examples:

(i) Let $f_n = 1/n\chi_{(0,n)}$. Then:
$f_n \to f$ in measure since $\{x : |f_n - 0| \geq \epsilon\} = \emptyset$ for $1/n < \epsilon$.

(iii) Let $f_n = n\chi_{[0,1/n]}$. Then:
$f_n \to 0$ in measure since $\{x : |f_n(x) - 0| \geq \epsilon\} \subseteq [0, 1/n]$.

(iv) Let $f_n = \chi_{[j/2^k,(j+1)/2^k]}$ where $n = 2^k + j$ and $0 \leq j \leq 2^k$. That is,
$f_1 = \chi_{[0,1]}, f_2 = \chi_{[0,1/2]}, f_3 = \chi_{[1/2,1]}, f_4 = \chi_{[0,1/4]}, f_5 = \chi_{[1/4,1/2]}$
$f_6 = \chi_{[1/2,3/4]}, f_7 = \chi_{[3/4,1]}, f_8 = \chi_{[0,1/8]}, \cdots$ Then:
$f_n \to 0$ in measure.

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Proposition 2.29: If $\int_X |f_n - f| d\mu \to 0$, that is $f_n \to f$ in L^1, then $f_n \to f$ in measure.

Proof: Given $\epsilon > 0$, let $E_{n,\epsilon} = \{x : |f_n - f| \geq \epsilon\}$. Then, $0 \leftarrow \int_X |f_n - f| d\mu \geq \int_{E_{n,\epsilon}} |f_n - f| d\mu \geq \epsilon\mu(E_{n,\epsilon}) \Rightarrow \lim_n \mu(E_{n,\epsilon}) = 0$ for all $\epsilon > 0$.

<u>Note</u> that the converse of Proposition 2.29 is false: Let $f_n = 1/n\chi_{(0,n)}$. Then, $f_n \to 0$ in measure, but $\int_{\mathbb{R}} |f_n - 0| dm = 1 \not\to 0$.

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Theorem 2.30: Let $(X, \mathcal{M}, \mu)$ be a measure space, and $f_n : X \to \mathbb{R}$ be measurable. Then, if $f_n \to f$ in measure and $f_n \to g$ in measure, then $f = g$ *a.e.* If $\{f_n\}$ is Cauchy in measure, then there exists $f : X \to \mathbb{R}$ which is measurable such that $f_n \to f$ in measure. Moreover, if $f_n \to f$ in measure, then there exists a subsequence $\{f_{n_j}\}$ such that $f_{n_j} \to f$ pointwise *a.e.*

[Note that $f_n \to f$ in measure $\Leftrightarrow \{f_n\}$ is Cauchy in measure.]

Proof: Let $(X, \mathcal{M}, \mu)$ be a measure space, and $f_n : X \to \mathbb{R}$ be measurable. Suppose that $f_n \to f$ in measure and $f_n \to g$ in measure. Then, given $\epsilon > 0$, $\{x : |f(x) - g(x)| \geq \epsilon\} \subseteq \{x : |f(x) - f_n(x)| \geq \epsilon/2\} \cup \{x : |g(x) - f_n(x)| \geq \epsilon/2\}$ because $\epsilon \leq |f(x) - g(x)| \leq |f(x) - f_n(x)| + |f_n(x) - g(x)|$. So, $\mu(\{x : |f(x) - g(x)| \geq \epsilon\}) \leq \mu(\{x : |f(x) - f_n(x)| \geq \epsilon/2\}) + \mu(\{x : |g(x) - f_n(x)| \geq \epsilon/2\})$ for all n, and also by the hypotheses $\mu(\{x : |f(x) - f_n(x)| \geq \epsilon/2\}) \to 0$ and $\mu(\{x : |g(x) - f_n(x)| \geq \epsilon/2\}) \to 0$ as $n \to \infty$.
Thus, $\mu(\{x : |f(x) - g(x)| \geq \epsilon\}) = 0$ for all $\epsilon > 0$.

Now, $\{x : f(x) \neq g(x)\} = \bigcup_{n=1}^{\infty} \{x : |f(x) - g(x)| \geq 1/n\} \Rightarrow \mu(\{x : f(x) \neq g(x)\}) = 0$. Thus, $f = g$ *a.e.*