HW10 Let $F: \mathbb{R} \to \mathbb{R}$ increasing and right continuous, $(\mathbb{R}, \mathcal{M}_F, \mu_F)$ the Lebesgue-Stieltjes measure.

Given $E \in \mathcal{M}_F$ with $\mu_F(E) < +\infty$

Prove that given $\epsilon > 0$ there exists $A = \bigcup_{j=1}^{n} (a_j, b_j)$

with $\mu_F(E \Delta A) < \epsilon$.

($E \Delta A = (E \setminus A) \cup (A \setminus E)$
this is called the symmetric difference of sets)

Proof of (b): Proposition 2.16, $\int |f-g|d\mu = 0 \Leftrightarrow |f-g| = 0$ $a.e.$ Also, $|f-g| = 0$ $a.e. \Leftrightarrow f = g$ $a.e.$ Thus, $\int |f-g|d\mu = 0$ $\Leftrightarrow f = g$ $a.e.$ Now, let $E \in \mathcal{M}$. Then, $\int |f-g|d\mu = 0, \Leftrightarrow$ $\int_E |f-g|d\mu = 0$. But, $0 \le |\int_E (f-g)d\mu| \le \int_E |f-g|d\mu$ $= 0$. Thus, $\int_E (f-g)d\mu = 0$, and so $\int_E f d\mu = \int_E g d\mu$ for all $E \in \mathcal{M}$. Next, suppose that $\int_E f d\mu = \int_E g d\mu$ for all $E \in \mathcal{M}$. (Complex case) Then, $\int_E Ref d\mu = \int_E Reg d\mu$ and $\int_E Imf d\mu$ $= \int_E Img d\mu$ for all $E \in \mathcal{M} \Rightarrow \int_E (Ref - Reg)d\mu = 0$ and $\int_E (Imf - Img)d\mu = 0$ for all $E \in \mathcal{M}$. Next, suppose that $E = \{x : Ref(x) - Reg(x) \ge 0\}$. Then, $\int_E (Ref - Reg)d\mu$ $= \int_E (Ref - Reg)^+ d\mu$. Therefore, by Proposition 2.16, $\mu(\{x \in E : Ref(x) - Reg(x) > 0\}) = 0$. Also, suppose that $E^c = \{x : Ref(x) - Reg(x) < 0\}$. Then, $\mu(E^c) = 0$. Thus, $Ref = Reg$ $a.e.$ Similarly, $Imf = Img$ $a.e.$, and so $f = g$ $a.e.$

<u>Remark</u>: Let $f : X \to \overline{\mathbb{R}}$ be measurable and $\int |f|d\mu < +\infty$. Then, $\int f^+ d\mu < +\infty$ and $\int f^- d\mu < +\infty \Rightarrow \mu(\{x : f^+(x) = +\infty\}) = 0$, $\mu(\{x : f^-(x) = +\infty\}) = 0$, $\mu(\{x : f^+(x) = -\infty\}) = 0$, and also $\mu(\{x : f^-(x) = -\infty\}) = 0$. So, up to a set of measure 0, f is real-valued. That is, $f : X \to \mathbb{R}$. "Throwing away the set of measure 0" does not effect integration.

Motivation: Define $f \sim g$ if and only if $f = g$ $a.e.$ Then, we see that $\sim$ is an equivalence relation. Now, let $\mathcal{N} = \{h : h = 0 \; a.e\} \subseteq \mathcal{L}^1(\mu)$. Then, $\mathcal{N}$ is a vector space, and $f = g$ $a.e.$ if and only if $g = f + h$ for some $h \in \mathcal{N}$. Also, cosets are $[f] = \{g : g \sim f\} = \{g : g = f \; a.e.\} = f + \mathcal{N}$.

Definition: $L^1(\mu) = \{[f] : f \in \mathcal{L}^1(\mu)\} = \mathcal{L}^1(\mu)/\mathcal{N}$

Proposition: Define $\rho([f],[g]) = \int |f-g|d\mu$. Then, ρ is well-defined, and is a metric on $L^1(\mu)$.

Proof: Suppose that $f_1 \sim f_2$ and $g_1 \sim g_2$. Then, $f_1 - g_1 \sim f_2 - g_2$, and so $f_1 - g_1 = f_2 - g_2$ $a.e.$ $\Rightarrow \int |f_1 - g_1|d\mu = \int |f_2 - g_2|d\mu$. Thus, ρ is well-defined. It is clear that $\rho([f],[g]) \ge 0$. Thus, ρ is nonnegative. Also, $\rho([f],[g]) = 0 \Leftrightarrow \int |f-g|d\mu = 0 \Leftrightarrow$ $f = g$ $a.e.$ (by Proposition 2.23) $\Leftrightarrow f \sim g \Leftrightarrow [f] = [g]$.

Next, $\rho([f],[g]) = \int |f-g|d\mu = \int |g-f|d\mu = \rho([g],[f])$. Thus, ρ is symmetric. Finally, given $[f],[g],[h] \in L^1(\mu)$, $\rho([f],[g]) = \int |f-g|d\mu = \int |(f-h)+(h-g)|d\mu \leq \int (|f-h|+|h-g|)d\mu = \int |f-h|d\mu + \int |h-g|d\mu = \rho([f],[h]) + \rho([h],[g])$. Thus, the triangular inequality holds. Thus, ρ is a metric on L^1 (not on $\mathcal{L}^1$.)

<u>Note</u>: In this metric space, $[f_n] \to [f] \Leftrightarrow \rho([f_n],[f]) \to 0 \Leftrightarrow \int |f_n - f|d\mu \to 0$.

Theorem 2.24 (Lebesgue Dominated Convergence Theorem - DCT) Let $\{f_n\} \subseteq \mathcal{L}^1(\mu)$, $f_n \to f$ $a.e.$ and there exist $g \in \mathcal{L}^1(\mu)$ such that $|f_n(x)| \leq g(x)$ $a.e.$ Then, $f \in \mathcal{L}^1(\mu)$ and $\int f d\mu = \lim_n \int f_n d\mu$.

Proof: (Real case) Let $\{f_n\} \subseteq \mathcal{L}^1(\mu)$, $f_n \to f$ $a.e.$ and there exist $g \in \mathcal{L}^1(\mu)$ such that $|f_n(x)| \leq g(x)$ $a.e. \Leftrightarrow |f(x)| \leq g(x) \Rightarrow g(x) + f_n(x) \geq 0$ $a.e.$ and $g(x) - f_n(x) \geq 0$ $a.e.$, and also $g(x) + f(x) \geq 0$ $a.e.$ and $g(x) - f(x) \geq 0$ $a.e.$ By Fatou's Lemma, $\int (g+f)d\mu \leq \lim_n inf \int (g+f_n)d\mu \Rightarrow \int f d\mu + \int g d\mu \leq \int g d\mu + \lim_n inf \int f_n d\mu \Rightarrow \int f d\mu \leq \lim_n inf \int f_n d\mu$. Also, similarly $\int g d\mu - \int f d\mu = \int (g-f)d\mu \leq \lim_n inf \int (g-f_n)d\mu = \lim_n inf(\int g d\mu - \int f_n d\mu) = \int g d\mu - \lim_n sup \int f_n d\mu \Rightarrow -\int f_n d\mu \leq -\lim_n sup \int f_n d\mu \Rightarrow \int f_n d\mu \geq \lim_n sup \int f_n d\mu$. Thus, the limit exists, and $\lim_n \int f_n d\mu = \int f d\mu$.

Theorem 2.25: Suppose that $\{f_j\} \subseteq \mathcal{L}^1(\mu)$ and $\sum_{j=1}^{\infty} \int |f_j| d\mu < +\infty$. Then, $\sum_{j=1}^{\infty} f_j(x)$ converges $a.e.$ Also, if we let $f(x) = \sum_{j=1}^{\infty} f_j(x)$, then $\int f d\mu = \int \sum_{j=1}^{\infty} f_j d\mu = \sum_{j=1}^{\infty} \int f_j d\mu$.

Proof: Define $g(x) = \sum_{j=1}^{\infty} |f_j(x)|$. (Notice that $g(x) = +\infty$ sometimes.)

Then, $S_N(x) = \sum_{j=1}^{N} |f_j(x)| \Rightarrow S_N \nearrow g$. Thus, by the Monotone

Convergence Theorem, $\int g d\mu = \lim_N \int S_N d\mu = \lim_N \sum_{j=1}^{N} \int |f_j| d\mu$

$\sum_{j=1}^{\infty} \int |f_j| d\mu < +\infty$. So, $g(x) \geq 0$ and $\int g d\mu < +\infty$ implies that $\mu(\{x : g(x) - +\infty\}) - 0$. Now, let $N - \{x : g(x) - +\infty\}$.

Then, if $x \notin N$, then $g(x) < +\infty \Rightarrow$ if $x \notin N$, then $\sum_{j=1}^{\infty} |f_j(x)|$ $< +\infty \Rightarrow$ if $x \notin N$, then $f(x) = \sum_{j=1}^{\infty} f_j(x) < +\infty$ since absolute convergence $\Rightarrow$ convergence. Thus, $f(x) = \sum_{j=1}^{\infty} f_j(x)$ is finite $a.e.$

Finally, if $h_N(x) = \sum_{j=1}^{N} f_j(x)$, then $|h_N(x)| = \left|\sum_{j=1}^{N} f_j(x)\right| \leq$ $\sum_{j=1}^{\infty} |f_j(x)| = g(x)$ and $h_N(x) \to f(x)$ $a.e.$ Thus, by Dominated Convergence Theorem, $\int f d\mu = \lim_N \int h_N d\mu = \lim_N \int \sum_{j=1}^{N} f_j d\mu =$ $\lim_N \sum_{j=1}^{N} \int f_j d\mu = \sum_{j=1}^{\infty} \int f_j d\mu$.

Example (~~[illegible]~~): Let $f(x) = x^{-1/2}$ if $0 < x < 1$, $f(x) = 0$ otherwise. Let $\{r_n\}_{n=1}^{\infty}$ enumeration of the rationals, and set $g(x) = \sum_{n=1}^{\infty} 2^{-n} f(x - r_n)$. Then:

(a) $g \in \mathcal{L}^1(m)$, and in particular g is finite $a.e.$

(b) g is discontinuous at every point and unbounded on every rational interval.

Proof: $\int f dm = R\int_0^1 x^{-1/2} dx = \lim_{\epsilon \downarrow 0} R\int_\epsilon^1 x^{-1/2} dx = \lim_{\epsilon \downarrow 0} 2x^{1/2}]_\epsilon^1 =$ $\lim_{\epsilon \downarrow 0} (2 - 2\sqrt{\epsilon}) = 2$. So, $\int f dm = 2$.

Let $\{r_n\}$ enumerate all rationals. Let $g(x) = \sum_{n=1}^{\infty} 2^{-n} f(x - r_n)$.

Then, $\int g dm = \int \sum_{n=1}^{\infty} 2^{-n} f(x - r_n) dm = \sum_{n=1}^{\infty} \int 2^{-n} f(x - r_n) dm$ $= \sum_{n=1}^{\infty} 2^{-n} \int f(x - r_n) dm = \sum_{n=1}^{\infty} 2^{-n} \cdot 2 < +\infty$. Thus, $\int g dm$

$< +\infty$, and so g is finite $a.e.$Therefore, for almost all x, $\sum_{n=1}^{\infty} 2^{-n} f(x - r_n)$ converges and we have calculated its integral, and $g(x)$ is unbounded in the neighborhood of every rational numbers $\Rightarrow g(x)$ is discontinuous at every rational number.

Theorem 2.26: Let $(X, \mathcal{M}, \mu)$ be a measure space. If $f \in \mathcal{L}^1(\mu)$ and $\epsilon > 0$, then there exists $\phi = \sum a_j \chi_{E_j}$ which is integrable and simple such that $\int |f - \phi| d\mu < \epsilon$. [That is, simple functions are dense in $(L^1(\mu), \rho)$ which is a metric space.] If μ is Lebesgue-Stieltjes measure on $\mathbb{R}$, then the set E_j can be taken to be a finite union of open intervals. Moreover, in this case (Lebesgue-Stieltjes measure), there exists g which is a continuous function that vanishes outside of a bounded interval and $\int |f - g| d\mu < \epsilon$.

Proof (Real case): Write $f = f^+ - f^-$. Then, we have simple functions $\phi_n \nearrow f^+$ and $\psi_n \nearrow f^-$. Thus, by the Monotone Convergence Theorem, $\lim_n \int \phi_n d\mu = \int f^+ d\mu$ and $\lim_n \int \psi_n d\mu = \int f^- d\mu$. Thus, $\lim_n \int (f^+ - \phi_n) d\mu = 0$ and $\lim_n \int (f^- - \psi_n) d\mu = 0$. Next, look at $\int |f - (\phi_n - \psi_n)| d\mu = \int |f^+ - f^- - \phi_n + \psi_n| d\mu \leq \int [|f^+ - \phi_n| + |f^- - \psi_n|] d\mu \to 0$. Pick n_0 such that $\phi = \phi_{n_0} - \psi_{n_0} = \sum a_j \chi_{E_j}$.

Now, assume that μ is a Lebesgue-Stieltjes measure. Then by Proposition 1.20 (Littlewood's First Principle), there exists U_j, a finite union of open intervals, such that $\mu(E_j \triangle U_j)$ which is arbitrarily small. Let $\gamma = \sum_{j=1}^{N} a_j \chi_{U_j}$. Pick $\phi = \sum_{j=1}^{N} a_j \chi_{E_j}$ such that $\int |f - \phi| d\mu < \epsilon/2$. Then, $\int |f - \gamma| d\mu \leq \int (|f - \phi| + |\phi - \gamma|) d\mu < \epsilon/2 + \int |\sum_{j=1}^{N} a_j (\chi_{E_j} - \chi_{U_j})| d\mu \leq \epsilon/2 + \int \sum_{j=1}^{N} |a_j| |\chi_{E_j} - \chi_{U_j}| d\mu = \epsilon/2 + \sum_{j=1}^{N} |a_j| \mu(E_j \triangle U_j) < \epsilon/2 + \epsilon/2 = \epsilon$ by picking U_j such that $\mu(E_j \triangle U_j) < \epsilon/2(|a_j| + 1)N$. Finally, given a U_j which is a finite union of open intervals, χ_{U_j} is 1 on these open intervals. Pick g_j continuous such that $\int |\chi_{U_j} - g_j| d\mu$ is arbitrarily small. Let $g = \sum_{j=1}^{N} a_j g_j$ to get