Chapter 0 Riemann Integration vs. Lebesgue Integration

Measure Theory was developed to take care of **Riemann Integration's** shortcomings.

0.1 Review of Riemann Integration

Definition: Given a bounded function $f : [a, b] \to \mathbb{R}$ and a **partition** $P = \{a = x_0 < x_1 < \ldots < x_n = b\}$, let $\Delta x_i = x_i - x_{i-1}$, $M_i = sup$ $\{f(x) : x_{i-1} \leq x \leq x_i\}$ and $m_i = inf\,\{f(x) : x_{i-1} \leq x \leq x_i\}$. Then,

Upper Riemann Sum is $U(P, f) = \sum_{i=1}^{n} M_i \cdot \Delta x_i$ and

Lower Riemann Sum is $L(P, f) = \sum_{i=1}^{n} m_i \cdot \Delta x_i$

Upper Riemann Integral is $\overline{\int_a^b} f\,dx = \inf_P U(P, f)$ and

Lower Riemann Integral is $\underline{\int_a^b} f\,dx = \sup_P L(P, f)$

Definition: f is called **Riemann integrable** if $\overline{\int_a^b} f\,dx = \underline{\int_a^b} f\,dx =$ $\equiv \int_a^b f\,dx$.

Facts:
1. If f is continuous, then f is Riemann integrable.
2. If f is piecewise continuous, then f is Riemann integrable.
3. Sums and products of Riemann integrable functions are Riemann integrable.
4. f needs to be bounded to be Riemann integrable.

Example: Let $\{x_n\}$ be the set of rational numbers in $[a, b]$ (remember that the set of rational numbers or a subset of the set of rational numbers is countable.)

Define $f(x) = \begin{cases} 1 & \text{if } x \text{ is rational} \\ 0 & \text{if } x \text{ is irrational} \end{cases}$

Take any partition P, then

$$U(P, f) = \sum_{i=1}^{n} M_i \cdot \Delta x_i = \sum_{i=1}^{n} 1 \cdot \Delta x_i = b - a \text{ and}$$

$$L(P, f) = \sum_{i=1}^{n} m_i \cdot \Delta x_i = \sum_{i=1}^{n} 0 \cdot \Delta x_i = 0$$

Thus, $\overline{\int_a^b} f\,dx = \inf_P U(P, f) = b - a \neq 0 = \underline{\int_a^b} f\,dx = \sup_P L(P, f)$

and so, f is <u>not</u> Riemann integrable.

0.2 Five Shortcomings of Riemann Integration

1. Are there any other ways to characterize Riemann integrable functions besides the definition?
   There was no clear method for characterizing the functions that are Riemann integrable. No specific set of necessary and sufficient conditions which can be applied to a function to determine whether or not it is Riemann integrable.
2. Riemann integrable functions behave badly for pointwise limits. Pointwise limits of Riemann integrable functions may not be Riemann integrable.

   Let $\{x_n\}$ be a countable dense set, such as the rationals, on the interval $[a, b]$.

   Define $f_n = \begin{cases} 1 & \text{if } x = x_1, x_2, x_3, \ldots, x_n, \text{ first } n \text{ rationals} \\ 0 & \text{otherwise} \end{cases}$

   Then, $\overline{\int_a^b} f_n \, dx = \inf_P U(P, f_n) = 0 = \underline{\int_a^b} f_n \, dx = \sup_P L(P, f_n)$

   and so f_n is Riemann integrable for all n.

   But, $0 \leq f_1(x) \leq f_2(x) \leq \ldots$ and the pointwise limit is

   $$\lim_{n\to\infty} f_n(x) = f(x) = \begin{cases} 1 & \text{if } x \text{ is rational} \\ 0 & \text{if } x \text{ is irrational} \end{cases}$$

   Clearly, $f(x)$ is <u>not</u> Riemann integrable since the upper Riemann integral is $\overline{\int_a^b} f \, dx = \inf_P U(P, f) = b - a$ and the lower Riemann integral is $\underline{\int_a^b} f \, dx = \sup_P L(P, f) = 0$.

   Thus, each f_n is Riemann integrable, but the pointwise limit is not.
3. Riemann integration is awkward when the function is unbounded or the domain of the function is unbounded.
   In order to handle these cases using Riemann integration, it is necessary to introduce the improper integral.
   That is, if f is unbounded at c where $a < c < b$, we need to calculate $\lim_{\epsilon\downarrow 0} \int_a^{c-\epsilon} f \, dx$, $\lim_{\epsilon\downarrow 0} \int_{c+\epsilon}^b f \, dx$, and if these limits exist , then they allow us to define the improper Riemann integral of f

   Let $h(x) = \sum_{n=1}^{\infty} \dfrac{2^{-n}}{\sqrt{|x - x_n|}}$ where $\{x_n\}$ is an enumeration of rationals. Then, h is bounded at all irrationals and unbounded at all rationals. This function h can be integrated easily using Lebesgue integral.

4. Fundamental Theorem of Calculus

$$\frac{d}{dx}\int_a^x f(t)\,dt = f(x) \text{ and}$$

$$\int_a^b f'(t)\,dt = f(b) - f(a) \text{ usually}$$

Problem: The theorem is true if f is continuously differentiable. But, the theorem is true for exactly which functions besides the one mentioned above?

5. Given a set of Riemann integrable functions, define a metric by $d(f, g) = \int_a^b |f - g|\,dt$. Note that the metric space equipped with this metric is not a complete metric space if we only use Riemann integrable functions.

0.3 Riemann Integration vs. Lebesgue Integration

Key idea for Riemann Integration: Uses vertical strips

Partitions the domain

$m_i \cdot \triangle x_i \leq A_i \leq M_i \cdot \triangle x_i$

Key idea for Lebesgue Integration:

Partitions the range

$l(E_i) \cdot y_{i-1} \leq A_i \leq l(E_i) \cdot y_i$ where $E_i = \{x : y_{i-1} \leq f(x) \leq y_i\}$

We need the idea of "length" of very general subsets of $\mathbb{R}$ because, in general, E_i could be a wild subset of x-axis.

Chapter 1 Measures

1.1 Introduction

To carry out Lebesgue's idea of integral, first we need to measure lengths of very general subsets.
This idea leads to Measure Theory.
It turns out that to have a notion of length, we can't use all subsets of $\mathbb{R}$.
A reasonable collection of subsets turns out to be what we call σ-algebras.

1.2 Algebras and σ-algebras of Sets

Let X be a nonempty set, and $\mathcal{P}(X)$ denote the set of all subsets of X which is called a **power set of X**.

Definition: A nonempty collection of subsets of X, $\mathcal{A} \subseteq \mathcal{P}(X)$ is called an **algebra of sets on X** provided that:

(1) If $A_1, A_2, \ldots, A_n \in \mathcal{A}$, then $A_1 \cup A_2 \cup \ldots \cup A_n \in \mathcal{A}$.
(2) If $A \in \mathcal{A}$, then $A^c \in \mathcal{A}$.

Proposition: Let $\mathcal{A}$ be an algebra of sets on X, then:

(3) $\emptyset, X \in \mathcal{A}$
(4) If $A, B \in \mathcal{A}$, then $A \backslash B \in \mathcal{A}$
where $A \backslash B \equiv \{x : x \in A \text{ and } x \notin B\}$
(5) If $A_1, A_2, \ldots, A_n \in \mathcal{A}$, then $A_1 \cap A_2 \cap \ldots \cap A_n \in \mathcal{A}$.

Proof (3): Since $\mathcal{A}$ is nonempty, there exists $A \in \mathcal{A}$. Then, $A^c \in \mathcal{A}$ by the definition of algebra. Thurs, $A \cup A^c = X \in \mathcal{A} \Rightarrow \emptyset = X^c \in \mathcal{A}$.

Proof (4): Let $A, B \in \mathcal{A}$. Then, $A^c \in \mathcal{A}$ and so $A^c \cup B \in \mathcal{A} \Rightarrow$ $(A^c \cup B)^c = A \cap B^c = A \backslash B \in \mathcal{A}$.

Proof (5): [It is enough to show $A_1 \cap A_2 \in \mathcal{A}$, then use induction.]
By DeMorgan's law, $(A_1 \cap A_2)^c = A_1^c \cup A_2^c$. Now, $A_1, A_2 \in \mathcal{A}$ $\Rightarrow A_1^c, A_2^c \in \mathcal{A} \Rightarrow A_1^c \cup A_2^c \in \mathcal{A} \Rightarrow (A_1 \cap A_2)^c \in \mathcal{A} \Rightarrow$ $((A_1 \cap A_2)^c)^c = A_1 \cap A_2 \in \mathcal{A}$

Definition: Let X be a nonempty set, an algebra of sets $\mathcal{A}$ on X is called a **σ-algebra of sets on X** provided that whenever $\{A_n\}_{n=1}^{\infty}$ is a countable collection of sets in $\mathcal{A}$, then $\bigcup_{n=1}^{\infty} A_n \in \mathcal{A}$.